

A Short Introduction To MEREOLOGY

Björn Mäurer — Alexander Heußner

Faculty of Mathematics and Computer Science
University of Leipzig

Abstract. Some short introductory thoughts about mereology followed by the remark on their isomorphic relation to boolean algebras without first-element. For a closer analysis we sketch completeness and the atomistic view as examples for metaproperties under this isomorphism.

This text is mainly based on [Ridder] Chapter III.

Mereology has been considered as an important feature of philosophy and metamathematics for nearly 100 years, its origins lie in set theory and logic calculus. But first: What does this term denote?

Mereology is derived from from the greek word $\mu\acute{\epsilon}\rho\omicron\varsigma$, this means “part”. Therefore mereology is the branch of science of analyzing the relation between the part and the whole based on suitable logical systems. For a more detailed introduction, especially about the fundamentals and their relationship to the “nominalistic program”, see [Ridder]. Mereological systems are used to iron out the problems with the two differnt operators \in and \subseteq in set theory and for that reason are an ideal tool for modeling especially in “formal ontology” (for example: [GOL],[Appl.Ont.Lab.]).

1 Mereology and Boolean Algebra:

Definition 1. *let M be a non-empty set and $x, y, z, r, s, w \in M$, then an “ordinary system of MEREOLOGY” is determined by the following definitions and axioms:*

first we make some formal definitions:

<i>identity</i>	$\forall x, y (x = y \leftrightarrow x \preceq y \wedge y \preceq x)$
<i>real part</i>	$\forall x, y (x \prec y \leftrightarrow x \preceq y \wedge x \neq y)$
<i>crossing</i>	$\forall x, y (x \circ y \leftrightarrow \exists z (z \preceq x \wedge z \preceq y))$
<i>discreteness</i>	$\forall x, y (x \wr y \leftrightarrow \neg(x \circ y))$

a mereological system is then axiomized by:

<i>axiom 1</i>	$\forall x, y (x \preceq y \leftrightarrow \forall z (z \circ x \rightarrow z \circ y))$
<i>axiom 2</i>	$\forall x, y \exists z \forall w (w \circ z \leftrightarrow w \circ x \vee w \circ y)$
<i>axiom 3</i>	$\forall x, y (x \circ y \rightarrow \exists z \forall w (w \preceq z \leftrightarrow w \preceq x \wedge w \preceq y))$
<i>axiom 4</i>	$\forall x (\exists y (y \wr x) \rightarrow \exists z \forall w (w \preceq z \leftrightarrow w \wr x))$

*and some further definitions used later:*¹

<i>binary sum</i>	$\forall x, y (x + y =_{def} (jz) (\forall w (w \circ z \leftrightarrow w \circ x \vee w \circ y)))$
<i>bin. product</i>	$\forall x, y (x \cdot y =_{def} (jz) (\forall w (w \preceq z \leftrightarrow w \preceq x \wedge w \preceq y)))$ ²
<i>complement</i>	$\forall x (x' =_{def} (jz) (\forall w (w \preceq z \leftrightarrow w \wr x)))$

¹ the j operator is the “jota operator” and denotes: *there exists a unique* which is equal to a proof of the uniqueness and existance

² for existence: $\forall x, y (x \circ y \rightarrow x \cdot y =_{def} \dots)$

As mentioned before: for a historical approach to the upper axioms see [Ridder]. The basic idea is clear if you look at the axioms with a set-theoretically coined view: the first axiom is something like “a part shares its inner parts with the above whole”, then axiom two, three and four remind of set-theoretic \cup , \cap and complement $\bar{}$ operator.

For our comparison we use a boolean algebra $(B, \sqsubseteq, \sqcup, \sqcap, ', 0, 1)$ axiomized via a partial order and the existence of the supremum and infimum which is based on Tarski’s ideas (see [Ridder], p.172f). That way a scent of similarity is apparent for readers familiar with basic mathematics.

As we will not show the isomorphism between mereological systems and boolean algebras, we will present the fundamental result: *in a mereological system no “least element” exists* (like the null element in boolean algebras).

Proposition 1. *given an ordinary system of mereology defined as above*

$$\text{if } \text{card}(M) \geq 2 \text{ then } \neg \exists n \in M (\forall z \in M (n \preceq z))$$

Proof. There are at least two elements $x, y \in M$ with $x \neq y$, therefore $\neg(x \preceq y \wedge y \preceq x)$ with ‘DeMorgan’: $\neg(x \preceq y) \vee \neg(y \preceq x)$; we want to prove by contradiction, therefore: assume there exists at least one $n \in M$ with $\forall z \in M (n \preceq z)$, let this special n here be named \bar{n} with $\forall z (\bar{n} \preceq z)$; this is $\bar{n} \preceq x \wedge \bar{n} \preceq y$.

Case 1 $\neg(x \preceq y)$: because of the SPP³ there exists always a $\bar{r} \preceq x$ with $\bar{r} \not\preceq y$; but because of our assumption there must hold $\bar{n} \preceq \bar{r}$, **but** $\bar{n} \not\preceq y$ and so $\neg(\bar{n} \preceq y)$ which is contradictory

Case 2: analogously, only change x and y □

This result is the groundwork for the isomorphism between mereological systems and boolean algebras without least element. We are sorry not to show here the exact proof for the existence of this fundamental relation.

Back to our modeling idea: everything which can be described as a mereological system, can be described via the above isomorphism as a boolean algebra without least element and vice versa.

What is the importance of this result? Boolean algebras were introduced by the English mathematician George Boole (1815—1864) as a basic structure in algebraic logic. That is why we have a tremendous amount of results and applications of these algebras, which we can – with some minor changes – apply for mereological structures as well. This will be one of the main tasks in mereological research for the next years, for some basic approaches see [Ridder] chapters III and IV.

2 Further Aspects:

Let us take a look at some enhancements of this two algebraic structures, first the “completeness” and in addition to this the concept of atoms.

If you expand a boolean algebra (here: without least element) defined as above, including a finite sum and a finite product with their infinite counterparts, you get an “complete boolean algebra”. This implies: *for every (finite or infinite) subset of the basicset B of the boolean algebra (without least element) there exists a supremum and an infimum*

Formally this property is added to the axiom system via the following formulae ⁴ :

$$\begin{array}{ll} \text{A9 } \Sigma y (y \in S) \in B & \text{A10 } \Pi y (y \in S) \in B \\ \forall x \in S (x \sqsubseteq \Sigma y (y \in S)) & \forall x \in S (\Pi y (y \in S) \sqsubseteq x) \\ \forall z \in B (\forall x \in S (x \sqsubseteq z) \rightarrow \Sigma y (y \in S) \sqsubseteq z) & \forall z \in B (\forall x \in S (z \sqsubseteq x) \rightarrow z \sqsubseteq \Pi y (y \in S)) \\ \\ \text{A11 } \forall x \in B (x \sqcap \Sigma y (y \in S) = \Sigma (x \sqcap y) (y \in S)) & \\ \forall x \in B (x \sqcup \Pi y (y \in S) = \Pi (x \sqcup y) (y \in S)) & \end{array}$$

³ the **Strong Supplement Principle** in mereology originates from the transitivity of \preceq with the first axiom and the definitions of crossing and discreteness (see [Ridder] p.174 footnote 22)

⁴ for the first 8 axioms see Appendix A.4

Now we want to extend the mereological system in the same way, hoping the isomorphism also holds for this extended axiom systems.

$$\begin{array}{ll}
\text{general sum} & \Sigma y(y \in S) =_{def} (jz) (\forall x (x \circ z \leftrightarrow \exists r (r \in S \wedge x \circ r))) \\
\text{general product} & \Pi y(y \in S) =_{def} (jz) (\forall x (x \preceq z \leftrightarrow \forall r (r \in S \rightarrow x \preceq r))) \\
\text{axiom 5}^5 & \forall S \subseteq M (S \neq \emptyset \rightarrow \exists z \forall x (x \circ z \leftrightarrow \exists r (r \in S \wedge x \circ r))) \\
\text{axiom 6} & \forall S \subseteq M (\exists x \forall y (y \in S \rightarrow x \preceq y) \rightarrow \exists z \forall r (r \preceq z \leftrightarrow \forall s (s \in S \rightarrow r \preceq s)))
\end{array}$$

As expected, again this two enhanced algebraic systems are isomorphic, the proof of this is an extension to the proof of the foundational isomorphism.

Out of our modeling idea the question arises whether there exists a smallest part – an atom. Not touching on the very old philosophical and physical question of their existence, this idea is introduced formally:

$$\begin{array}{ll}
\text{boolean algebra:} & \forall x \in B (Atom(x) :\leftrightarrow x \neq 0 \wedge \forall y \in B (y \neq 0 \wedge y \sqsubseteq x) \rightarrow y = x)) \\
\text{mereological sys.:} & \forall x \in M (Atom(x) :\leftrightarrow \neg \exists y \in M (y \prec x))
\end{array}$$

And therefore a “atomistic boolean algebra” (without least element) and the mereological counterpart are easily formalized with the following additional axioms:

$$\begin{array}{ll}
\text{axiom BA:} & \forall x \in B (x \neq 0 \rightarrow \exists y \in B (Atom(y) \wedge y \sqsubseteq x)) \\
\text{axiom MA:} & \forall x \in M (\exists y \in M (Atom(y) \wedge y \preceq x))
\end{array}$$

Out of this sketch, other atomistic views are easily derived like an nonatomistic structure with or without atoms or atomless structures.

Besides the lack of the proof of some metaproperties (formal completeness...) for the above axiomatic systems, we state here again the existence of an isomorphism between the mereological structures and their counterparts in boolean algebra which, at this state, is more obvious to the reader when comparing the axioms literally.

To conclude, mereological systems are a basic tool in modeling and because of their strong relation to standard algebra can easily be derived therefrom and extended with important properties like atomism needed in formal ontology.

References

- [Ridder] Lothar Ridder: “Mereologie - ein Beitrag zur Ontologie und Erkenntnistheorie” in Philosophische Abhandlungen Band 83, Verlag Vittorio Klostermann Frankfurt 2002
[GOL] General Ontological Language: <http://www.ontology.uni-leipzig.de>
[Appl.Ont.Lab.] Laboratory for Applied Ontology: <http://ontology.ip.rm.cnr.it>

⁵ attention: this formula is not part of a first order language, it uses quantification over sets – to avoid this, it needs to be rewritten with an infinite structure of first order formulae

A more detailed View:

The above view is not sufficient for a mathematician's view onto the connection between boolean algebra and mereology, therefore first of all a better definition of the "boolean algebra without least element" is needed which leads to a more detailed analysis of the question: what is behind the generic term of the "isomorphism"? and thenceforwards a better sketch of the proof is easy to be drawn.

A.1 The "Quasi Boolean Algebra"

How can we imagine a boolean algebra without least element? and how to define the supremum and infimum then?

First of all the set $x \leq = \{y \mid x \leq y \leq 1\}$ for every chosen $x \in B$ defines a kind of boolean base set without zero (least element). Then a quasi boolean algebra is defined as a boolean algebra (see A.4) $(x \leq, \sqsubseteq, \sqcup, \sqcap, comp(x), 0_x, 1)$ with $comp(x)$ as the complement with reference to x and 0_x as the least element relative to x . This kind of algebraic structure is known as distributive half-lattices without least element.

With this method it is easy to derive a quasi boolean algebra from a "normal" boolean algebra.

A.2 The Isomorphism

Isomorphisms are basic to all branches of mathematics and often used, but why can we talk here about an isomorphism? Let us take a short review of fundamental mathematics...

We are comparing neither a boolean algebra and a mereological system nor some kind of their classification, the only thing to talk about are theories. Thus all we did above is to define two theories, but theories can not be isomorphic only their models can be. What we do is comparing a model $\mathfrak{A} \in Mod(Th(Mereology))$ and $\mathfrak{B} \in Mod(Th(BooleamAlgebra))$ and here we can say \mathfrak{A} is isomorphic to \mathfrak{B} .

A.3 A More Precise Sketch of the Proof ⁶

When talking about theories we must be aware of the underlying structures, the languages belonging to the upper two theories \mathcal{L}_{bool} and $\mathcal{L}_{mereol.}$. Now we can interspect interpretations of one language in the other. If such an interpretation exists both ways, or more informal: we can express the theorie of boolean algebra in the language of mereology and vice versa, we can conclude to the underlying isomorphism.

Definition 2.

a mapping $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ with two languages $\mathcal{L}_{1/2}$ is an interpretation iff for all $\psi, \phi \in \mathcal{L}_1$:

- (i) $Th(\mathcal{L}_1) \models \psi$ iff $Th(\mathcal{L}_2) \models f(\psi)$
- (ii) $Th(\mathcal{L}_1) \models \neg\psi$ iff $Th(\mathcal{L}_2) \models \neg f(\psi)$
- (iii) $Th(\mathcal{L}_1) \models \psi \vee \phi$ iff $Th(\mathcal{L}_2) \models f(\psi) \vee f(\phi)$
- (iv) $Th(\mathcal{L}_1) \models \psi \wedge \phi$ iff $Th(\mathcal{L}_2) \models f(\psi) \wedge f(\phi)$

The most intuitive approach to interpret mereology in (quasi) boolean algebra would be something like an interpretation f :

$$f : Th(\{\sqcup, \sqcap, =, \sqsubseteq, '\}) \rightarrow Th(\{*, +, =, \preceq, '\}) \quad ^7$$

To proof the isomorphism we have to interpret all the axioms of boolean algebra (except for those which define the zero-element) in mereology and proof the results against the mereological axioms and definitions of section 1.

⁶ "How precise can a sketch be?"

⁷ here: consider the two sets as ordered sets, this means: $f : \sqsubseteq \mapsto \preceq$ and so on...

A.4 Boolean Algebra revisited

let x, y and z be elements of the set B , therefore $(B, \sqsubseteq, \sqcup, \sqcap, ', 0, 1)$ – the ordinary system of “boolean algebra” – is described by the following axioms: (see [Ridder] p. 172f)

$$\begin{array}{ll}
 \text{A1 } \forall x (x \sqsubseteq x) & \text{A2 } \forall x, y, z ((x \sqsubseteq y \wedge y \sqsubseteq x) \rightarrow x \sqsubseteq y) \\
 & \text{A3 } \forall x, a (x = y \leftrightarrow (x \sqsubseteq y \wedge y \sqsubseteq x)) \\
 \\
 \text{A4 } \forall x, y (x \sqcup y \in B) & \text{A5 } \forall x, y (x \sqcap y \in B) \\
 \quad \forall x, y (x \sqsubseteq x \sqcup y \wedge y \sqsubseteq x \sqcup y) & \quad \forall x, y (x \sqcap y \sqsubseteq x \wedge x \sqcap y \sqsubseteq y) \\
 \quad \forall x, y, z ((x \sqsubseteq z \wedge y \sqsubseteq z) \rightarrow x \sqcup y \sqsubseteq z) & \quad \forall x, y, z ((z \sqsubseteq x \wedge z \sqsubseteq y) \rightarrow z \sqsubseteq x \sqcap y) \\
 \\
 \text{A6 } \forall x, y, z (x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)) & \quad \forall x, y, z (x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)) \\
 \\
 \text{A7 } 0, 1 \in B, \quad \forall x (0 \sqsubseteq x \wedge x \sqsubseteq 1) & \quad \text{A8 } \forall x (x' \in B), \forall x (x \sqcap x' = 0), \forall x (x \sqcup x' = 1)
 \end{array}$$

A.5 Proof

Because of the lack of a usable axiomatization of the system of quasi boolean algebra, only the direction from mereology to (quasi) boolean algebra is shown. for the axiomatization of the “target” of the upper mapping f , the upper axiom system is used, which is strongly correlated to the lacking system of the quasi boolean algebra because of the construction ideas in A.1. Without proof, we assume, that the upper axioms hold in the system of quasi boolean algebra. The second direction of the proof of the “isomorphism” is furthermore withheld to the mathematical reader.

- $f : A1 \mapsto \forall x (x \preceq x)$
 out of *axiom1* with $x =_{def} y : \forall x (x \preceq x \leftrightarrow \forall z (z \circ x \rightarrow z \circ x))$
 the last part is tautological, therefore $\forall x (x \preceq x)$ holds
- $f : A2 \mapsto \forall x, y, z ((x \preceq y \wedge y \preceq z) \rightarrow x \preceq z)$
 first we substitute the \preceq by using *axiom1*, then:

$$\left. \begin{array}{l}
 x \preceq y \leftrightarrow \forall a (a \circ x \rightarrow a \circ y) \\
 y \preceq z \leftrightarrow \forall a (a \circ y \rightarrow a \circ z)
 \end{array} \right\} \Rightarrow \forall a (a \circ x \rightarrow a \circ z) \Leftrightarrow x \preceq z$$

- $f : A3 \mapsto \forall x, y (x = y \leftrightarrow (x \preceq y \wedge y \preceq x))$
 this is exactly the definition of identity which is defined under the first case above (*identity*)
- $f : A4.1 \mapsto \forall x, y (x + y)$
 from definition (*binarysum*) we know $x + y =_{def} (jz)(\forall (w(w \circ z \leftrightarrow w \circ x \vee w \circ y)))$ this definition is not contradictory because existence follows directly by (*axiom2*) – for the uniqueness we assume that z_1 and z_2 are different and that way:

$$\left. \begin{array}{l}
 x + y = z_1 \implies \forall w (w \circ z_1 \leftrightarrow w \circ x \vee w \circ y) \\
 x + y = z_2 \implies \forall w (w \circ z_2 \leftrightarrow w \circ x \vee w \circ y)
 \end{array} \right\} \implies \forall w (w \circ z_1 \leftrightarrow w \circ z_2)$$

this implies $(w \circ z_1 \rightarrow w \circ z_2) \leftrightarrow z_1 \preceq z_2$ and $(w \circ z_2 \rightarrow w \circ z_1) \leftrightarrow z_2 \preceq z_1$
 it follows $z_1 = z_2$, therefore $f(A4.1)$ holds.

- $f : A4.2 \mapsto \forall x, y (x \preceq (x + y) \wedge y \preceq (x + y))$
 the definition (*binary sum*) implies $\forall w (w \circ (x + y) \leftrightarrow w \circ x \vee w \circ y)$; we assume that $w \circ x$ holds, likewise $(w \circ x \vee w \circ y)$ holds, but this is equivalent to $w \circ (x + y)$;

$$\forall w (w \circ x \rightarrow w \circ (x + y)) \stackrel{(axiom1)}{\implies} x \preceq (x + y)$$

the case $y \preceq (x + y)$ is shown analogously; so $f(A4.2)$ holds.

- $f : A4.3 \mapsto \forall x, y, z((x \preceq z \wedge y \preceq z) \rightarrow x + y \preceq z)$
which is *axiom1* insert to the definition of $x + y$
- $f : A5.1 \mapsto \forall x, y (x \circ y \rightarrow x * y)$
the next axiom is more difficult because in mereology the zero-element does not exist; to get an interpretable result we have to add the restriction $x \circ y$. to all images of axiom A5 except A5.3.
from definition (*binaryproduct*) we know $\forall x \circ y(x * y =_{def} (jz)(\forall w(w \preceq z \leftrightarrow w \preceq x \wedge w \preceq y)))$ similar to $f(A4.1)$ we have to show, that the definition is immaculately; the existence is obvious, because of *axiom3*; for the uniqueness we assume $z_1 = x * y$ and $z_2 = x * y$.
therefore: $\forall w : (w \preceq z_1 \leftrightarrow w \preceq x \wedge w \preceq y) \leftrightarrow (w \preceq z_2 \leftrightarrow w \preceq x \wedge w \preceq y) \leftrightarrow z_2 = z_1$
- $f : A5.2 \mapsto \forall x, y (x \circ y) \rightarrow (x * y \preceq x \wedge x * y \preceq y)$
after inserting $w =_{def} z$ into the definition of $x * y$ one gets $x * y \preceq x * y \leftrightarrow x * y \preceq x \wedge x * y \leftrightarrow y$; since the first part is tautological the assertion is true.
- $f : A5.3 \mapsto \forall x, y (x \circ y) \rightarrow \forall z((z \preceq x \wedge z \preceq y) \rightarrow z \preceq x * y)$
this is *axiom1* inserted directly into the definition of $x * y$
- $f : A6.1 \mapsto \forall x, y, z(x \circ y) \wedge (x \circ z) (x * (y + z) = (x * y) + (x * z))$

out of the earlier definitions:

$$x * (y + z) = x * \underbrace{(j\psi)(\forall(w \circ \psi \leftrightarrow w \circ y \vee w \circ z))}_{\psi} = (j\phi)(\forall v(v \preceq \phi \leftrightarrow v \preceq x \wedge v \preceq \psi)) \text{ and}$$

$$(x * y) + (x * z) = ((j\alpha)(\forall m(m \preceq \alpha \leftrightarrow m \preceq x \wedge m \preceq y))) + ((j\beta)(\forall n(n \preceq \beta \leftrightarrow n \preceq x \wedge n \preceq z)))$$

since we know of the uniqueness of “+” and “*” combined with the condition of the existence of $x \circ y$:

$$\forall w(w \circ (y + z) \leftrightarrow w \circ y \vee w \circ z) \forall v(v \preceq (x * (y + z)) \leftrightarrow v \preceq x \wedge v \preceq (y + z))$$

let v be part of $x * (y + z)$, this is:

$$\begin{aligned} v \preceq x * (y + z) &\Leftrightarrow v \preceq x \wedge v \preceq (y + z) \Leftrightarrow \forall \alpha : \alpha \circ v \rightarrow \alpha \circ x, \alpha \circ (y + z) \Leftrightarrow \alpha \circ x \wedge (\alpha \circ y \vee \alpha \circ z) \\ &\Leftrightarrow \forall \alpha : \alpha \circ v \rightarrow \exists \beta(\beta \preceq \alpha \wedge \beta \preceq x \wedge (\beta \preceq y \vee \beta \preceq z)) \\ &\Leftrightarrow \forall \alpha : \alpha \circ v \rightarrow \exists \beta(\beta \preceq \alpha \wedge \beta \preceq (x * y) \vee \beta \preceq (x * z)) \\ &\Leftrightarrow \forall \alpha : \alpha \circ v \rightarrow \alpha \circ (x * y) \vee \alpha \circ (x * z) \\ &\Leftrightarrow \forall \alpha : \alpha \circ v \rightarrow \alpha \circ ((x * y) + (x * z)) \\ &\Leftrightarrow v \preceq (x * y) + (x * z) \end{aligned}$$

- $f : A6.2 \mapsto \forall x, y, z((x + y) \circ (x + z) \wedge (y \circ z) : x + (y * z) = (x + y) * (x + z))$
analogously to A6.1 ...
- in A7 the zero does not make sense because of the lack of the corresponding element in standard-mereology. (remark: in quasi boolean algebras does not exist any zero-element but a least element!)
- $f : A7.1 \mapsto 1$, but first we have to define a 1 in mereology, therefore it is to be proven that:
 $\forall x, y \in \text{“mereology”} (x + x' = y + y')$.

$$\forall w : w \preceq x' \leftrightarrow w \wedge x \Leftrightarrow \forall w : w \preceq x' \leftrightarrow \neg w \circ x \Rightarrow \forall w : w \circ (x + x') \leftrightarrow w \circ x \vee w \circ x'$$

since x' is the complement: $\neg(w \circ x) \rightarrow w \preceq x' \rightarrow w \circ x'$. means, therefore: $\forall w(w \circ x + x')$

out of *axiom1*: $\forall w(w \preceq x + x')$ and thus $\forall x, y(x + x' = y + y')$, now we define $x + x'$ as 1 in mereological context.

$f(A7.1)$ is therefore a trivium and $f : A7.2 \mapsto \forall x(x \preceq 1)$ is already shown

- $f(A8.1)$ is verified by the definition of the *complement*
- $f(A8.2)$ does not make sense, because of the missing zero-element
- $f(A8.3)$ is verified by our definition of 1 in mereology