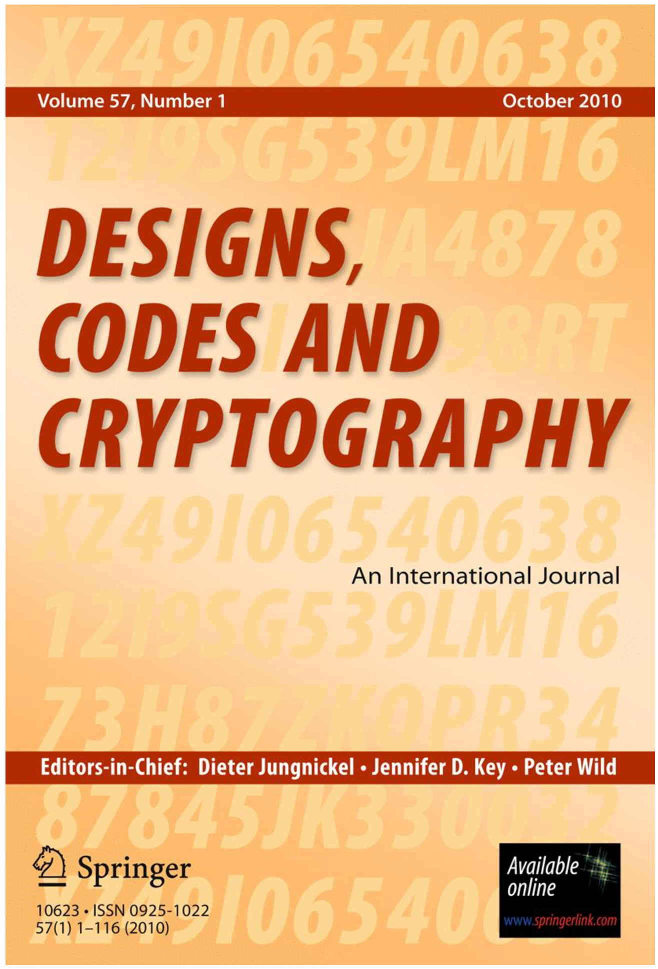
ISSN 0925-1022, Volume 57, Number 1



Des. Codes Cryptogr. (2010) 57:45–
DOI 10.1007/s10623-009-9348-1

On uniformly resolvable designs with block sizes 3 and 4

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Received: 9 December 2008 / Revised: 2 April 2009 / Accepted: 17 November 2009 /
Published online: 13 December 2009

Abstract A Uniformly Resolvable Design (URD) is a resolvable design in which each parallel class contains blocks of only one block size k, such a class is denoted k -pc and for a given k the number of k -pcs is denoted r_k. In this paper we consider the case of block sizes 3 and 4. The cases $r_3 = 1$ and $r_4 = 1$ correspond to Resolvable Group Divisible Designs (RGDD). We prove that if a 4-RGDD of type h^u exists then all admissible $\{3, 4\}$-URDs with $12hu$ points exist. In particular, this gives existence for URD with $v \equiv 0 \pmod{48}$ points. We also investigate the case of URDs with a fixed number of k -pc. In particular, we show that URDs with $r_3 = 4$ exist, and that those with $r_3 = 7, 10$ exist, with 11 and 12 possible exceptions respectively, this covers all cases with $1 < r_3 \leq 10$. Furthermore, we prove that URDs with $r_4 = 7$ exist and that those with $r_4 = 9$ exist, except when $v = 12, 24$ and possibly when $v = 276$. In addition, we prove that there exist 4-RGDDs of types 2^{142}, 2^{346} and 6^{54}. Finally, we provide four {3,5}-URDs with 105 points.

Keywords Uniformly resolvable design · Labeled uniformly resolvable design · Resolvable group divisible design · Frame · Transversal design

Mathematics Subject Classification (2000) 05B05 · 05B07

Communicated by Charles J Colbourn.

Electronic supplementary material The online version of this article (doi:10.1007/s10623-009-9348-1) contains supplementary material, which is available to authorized users.

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1 Introduction

Let v and λ be positive integers, let K and M be two sets of positive integers. A *group divisible design*, denoted $\mathrm{GDD}_\lambda(K, M; v)$, is a triple $(X, \boldsymbol{G}, \boldsymbol{B})$, where X is a set with v elements (called points), $\boldsymbol{G}$ is a set of subsets (called groups) of X, $\boldsymbol{G}$ partitions X, and $\boldsymbol{B}$ is a set of subsets (called blocks) of X such that

1. $|B| \in K$ for each $B \in \boldsymbol{B}$,
2. $|G| \in M$ for each $G \in \boldsymbol{G}$,
3. $|B \cap G| \leq 1$ for each $B \in \boldsymbol{B}$ and each $G \in \boldsymbol{G}$,
4. Each pair of elements of X from distinct groups is contained in exactly λ blocks.

The notation is similar to [3,4]. If $\lambda = 1$, the index λ is omitted. If $K = \{k\}$, respectively $M = \{m\}$, then the $\mathrm{GDD}_\lambda(K, M; v)$ is simply denoted $\mathrm{GDD}_\lambda(k, M; v)$ respectively $\mathrm{GDD}_\lambda(K, m; v)$, which is also specified in "exponential" form as $K\text{-}GDD_\lambda$ of type $m^{v/m}$. A $\mathrm{GDD}_\lambda(K, 1; v)$ is called a *pairwise balanced design* and denoted $\mathrm{PBD}_\lambda(K; v)$.

Theorem 1.1 ([18,23]) *There exists a* 4-GDD *of type* g^4m^1 *with* $m > 0$ *if, and only if,* $g \equiv m \equiv 0 \pmod 3$ *and* $0 < m \leq 3g/2$.

Theorem 1.2 ([1,15,23]) *There exists a* 5-GDD *of type* g^5m^1 *with* $m > 0$ *if* $g \equiv m \equiv 0 \pmod 4$ *and* $0 < m \leq 4g/3$*, with the possible exceptions of* $(g, m) = (12, 4)$ *and* $(12, 8)$.

A *transversal design* $\mathrm{TD}_\lambda(k, g)$, is equivalent to a $\mathrm{GDD}_\lambda(k, g; kg)$. That means, each block in a $\mathrm{TD}_\lambda(k, g)$ contains a point from each group. If $\lambda = 1$, the index λ is omitted.

Theorem 1.3 ([2]) *A* $\mathrm{TD}(k, g)$ *exists in the following cases:*

1. $k = 6$ *and* $g \geq 5$ *and* $g \notin \{6, 10, 14, 18, 22\}$;
2. $k = 7$ *and* $g \geq 7$ *and* $g \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$.

In a $\mathrm{GDD}_\lambda(K, M; v)$ with $(X, \boldsymbol{G}, \boldsymbol{B})$, a *parallel class* is a set of blocks, which partitions X. If $\boldsymbol{B}$ can be partitioned into parallel classes, then the $\mathrm{GDD}_\lambda(K, M; v)$ is said to be *resolvable* and denoted $\mathrm{RGDD}_\lambda(K, M; v)$. Analogously, a resolvable $\mathrm{PBD}_\lambda(K; v)$ is denoted $\mathrm{RPBD}_\lambda(K; v)$. A parallel class is said to be *uniform* if it contains blocks of only one size k (k-pc). If all parallel classes of an $\mathrm{RPBD}_\lambda(K; v)$ are uniform, the design is said to be *uniformly resolvable*. Here, a uniformly resolvable design $\mathrm{RPBD}_\lambda(K; v)$ is denoted $\mathrm{URD}_\lambda(K; v)$. If $\lambda = 1$, the index λ is omitted. In a $\mathrm{URD}_\lambda(K; v)$ the number of resolution classes with blocks of size k is denoted r_k, $k \in K$. Uniformly resolvable designs with block sizes 3 and 4 mean here $\mathrm{URD}(\{3, 4\}; v)$ with $r_3 > 0$ and $r_4 > 0$.

The following theorem about RGDDs will be applied later.

Theorem 1.4 ([4,10–14,17,24,28,30,31]) *The necessary conditions for the existence of an* k *-RGDD of type* h^n, $\mathrm{RGDD}(k, h; hn)$*, namely,* $n \geq k$, $hn \equiv 0 \pmod k$ *and* $h(n-1) \equiv 0 \pmod{k-1}$*, are also sufficient for*
$k = 2$;
$k = 3$, *except for* $(h, n) \in \{(2, 3), (2, 6), (6, 3)\}$; *and for*
$k = 4$, *except for* $(h, n) \in \{(2, 4), (2, 10), (3, 4), (6, 4)\}$ *and possibly excepting:*

1. $h \equiv 2,\ 10 (\mathrm{mod}\ 12)$:
   $h = 2$ *and* $n \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 142, 178, 184, 202, 214, 238, 250, 334, 346\}$;
   $h = 10$ *and* $n \in \{4, 34, 52, 94\}$;
   $h \in [14, 454] \cup \{478, 502, 514, 526, 614, 626, 686\}$ *and* $n \in \{10, 70, 82\}$.
2. $h \equiv 6 (\mathrm{mod} 12)$: $h = 6$ *and* $n \in \{6, 54, 68\}$; $h = 18$ *and* $n \in \{18, 38, 62\}$.
3. $h \equiv 9 (\mathrm{mod} 12)$: $h = 9$ *and* $n = 44$.
4. $h \equiv 0 (\mathrm{mod} 12)$: $h = 36$ *and* $n \in \{11, 14, 15, 18, 23\}$.

A *resolvable transversal design* $\mathrm{RTD}_\lambda(k, g)$, is equivalent to an $\mathrm{RGDD}_\lambda(k,\ g;\ kg)$. That means, each block in an $\mathrm{RTD}_\lambda(k, g)$ contains a point from each group. A K *-frame* is a GDD $(X,\ \boldsymbol{G},\ \boldsymbol{B})$ with index unity, in which the collection of blocks $\boldsymbol{B}$ can be partitioned into holey parallel classes each of which partitions $X \backslash G$ for some $G \in \boldsymbol{G}$. We use the usual exponential notation for the types of GDDs and frames. Thus, a GDD or a frame of type $1^i 2^j \ldots$ is one in which there are i groups of size 1, j groups of size 2, and so on. A K -frame is called *uniform* if each partial parallel class is of only one block size. It is called *completely uniform* if for each hole G the resolution classes which partitions $X \backslash G$ are all of one block size. We use mostly $K = \{3,\ 4\}$. A $\{3,\ 4\} -$ frame of type $(g; 3^{n_1} 4^{n_2})^u (m; 3^{n_3} 4^{n_4})^1$ has u groups of size g. Each group of size g has n_1 holey pcs of block size 3 and n_2 holey pcs of block size 4. The only group of size m has n_3 holey pcs of block size 3 and n_4 holey pcs of block size 4.

Theorem 1.5 ([24]) *For* $k = 2$ *and* $k = 3$ *there exists a* k *-frame of type* h^u *if, and only if,* $u \geq k + 1$, $h \equiv 0 \ (\mathrm{mod}\ k - 1)$, *and* $h \cdot (u - 1) \equiv 0 (\mathrm{mod}\ k)$.

Theorem 1.6 ([9,14,16,17,20,24,32]) *There exists a 4-frame of type* h^u *if, and only if,* $u \geq 5$, $h \equiv 0 \ (\mathrm{mod}\ 3)$ *and* $h \cdot (u - 1) \equiv 0 \ (\mathrm{mod}\ 4)$, *except possibly where*

1. $h = 36$ *and* $u = 12$;
2. $h \equiv 6 \ (\mathrm{mod}\ 12)$:
   $h = 6$ *and* $u \in \{7, 23, 27, 35, 39, 47\}$;
   $h = 18$ *and* $u \in \{15, 23, 27\}$;
   $h \in \{30\} \cup [66, 2190]$ *and* $u \in \{7, 23, 27, 39, 47\}$;
   $h \in \{42, 54\} \cup [2202, 11238]$ *and* $u \in \{23, 27\}$.

Later on some incomplete group divisible designs are applied. An *incomplete group divisible design* (IGDD) with block sizes from a set K and index unity is a quadruple $(X, \boldsymbol{G}, H, \boldsymbol{B})$, which satisfies the following properties:

1. $\boldsymbol{G} = \{G_1, G_2, \ldots, G_n\}$ is a partition of the set X of points into subsets called *groups*,
2. H is a subset of X called the *hole*,
3. $\boldsymbol{B}$ is a collection of subsets of X with cardinalities from K, called *blocks*, so that a group and a block contain at most one common point,
4. every pair of points from distinct groups is either in H or occurs in a unique block but not both.

This design is denoted by $\mathrm{IGDD}(K, M;\ v)$ of type T, where $M = \{|G_1|, |G_2|, \ldots, |G_n|\}$ and T is the multiset $\{(|G_i|,\ |G_i \cap H|) : 1 \leq i \leq n\}$. Sometimes "exponential" notation is used to describe the type. An $\mathrm{IGDD}(K,\ M;\ v)$ of type T is said to be *uniformly resolvable* and denoted by $\mathrm{IUGDD}(K,\ M;\ v)$ of type T if blocks can be partitioned into uniform parallel classes and partial uniform parallel classes, the latter partitioning $X \setminus H$. The numbers of uniform parallel classes, partial uniform parallel classes with blocks of size k are denoted by r_k, r_k°, respectively. If $|G_i| = 1$ for $1 \le i \le n$, then the IUGDD is denoted *incomplete uniformly resolvable design* IURD$(K; v)$ with a hole H.

Some known results about URDs are summarized below. Rees [21] introduced URDs and showed:

Theorem 1.7 ([21]) *There exists a* URD$(\{2, 3\}; v)$ *with* $r_2, r_3 > 0$ *if, and only if,*

1. $v \equiv 0 (\mathrm{mod}\ 6)$;
2. $r_2 = v - 1 - 2r_3 \quad \left(r_3 = \frac{v-1-r_2}{2}\right)$;
3. $1 \le r_3 \le \frac{v}{2} - 1$;

with the two exceptions $(v, r_3) = (6, 2),\ (12, 5)$.

Recently, almost all URDs with $K = \{2, 4\}$ were constructed in [7] which we improve slightly as follows:

Theorem 1.8 *There exists a* URD$(\{2, 4\}; v)$ *with* $r_2, r_4 > 0$ *if, and only if,*

1. $v \equiv 0 (\mathrm{mod}\ 4)$;
2. $r_2 = v - 1 - 3r_4 \quad \left(r_4 = \frac{v-1-r_2}{3}\right)$;

with two exceptions $(v, r_2) = (8, 1),\ (20, 1)$ *and possibly excepting:*

$(v, r_2) = (2n, 1), \quad n \in \{52, 100, 184\}$;
$(v, r_2) = (2n, r_2), \quad n \in \{34, 46, 70, 82, 94, 118, 130, 178, 202, 214, 238, 250, 334\}, \quad r_2$ *admissible;*
$(v, r_2) = (12n, 2), \quad n \in N = \{2, 7, 9, 10, 11, 13, 14, 17, 19, 22, 31, 34, 38, 43, 46, 47, 82\}$.

Proof Because each 4-pc can be replaced by three 2-pcs, only URDs with minimal r_2 are needed. In each $\{2, 4\}$ - URD there is $v \equiv 0, 4$ or 8 (mod 12) as the number of points.

For $v \equiv 4$ (mod12) there exists an RPBD$(4; v)$ by [19] or Theorem 1.4. That means that the minimal r_2 is zero.

For $v \equiv 8$ (mod 12) there exists a 4 -RGDD of type $2^{v/2}$ by Theorem 1.4, possibly excepting the above values. The 4-RGDDs of types 2^{142} and 2^{346} are given in Theorem 10.1 and 10.2. We only have to show that a URD$(\{2, 4\}; v)$ with $r_2 = 4$ exists for $v \in \{104, 200, 368\}$. There exists a 4 -RGDD of type 8^{3i+1} by Theorem 1.4. By filling all groups with a 2 -RGDD of type 4^2, which exists by Theorem 1.4, we obtain a URD$(\{2, 4\}; 24i + 8)$ with $r_2 = 4$ and the desired designs for $i \in \{4, 8, 15\}$.

For $v \equiv 0$ (mod 12) in [7] it is shown that a URD$(\{2, 4\}; v)$ with $r_2 = 2$ exists, possibly excepting the values from N. There exists a 4 -RGDD of type 6^{2i} for $i \in \{N - \{2, 34\}\}$ by Theorem 1.4. By filling all groups with a RPBD$(2; 6)$, which exists by Theorem 1.4, we obtain a URD$(\{2, 4\}; 12i)$ with $r_2 = 5$ for $i \in \{N - \{2, 34\}\}$. A URD$(\{2, 4\}; 24)$ with $r_2 = 5$ is contained in the Online Resource. There exists a 4 -RGDD of type 24^{17} by Theorem 1.4. By filling all groups with URD$(\{2, 4\}; 24)$ with $r_2 = 5$, we obtain a URD$(\{2, 4\}; 12 \cdot 34)$ with $r_2 = 5$. □

Theorem 1.9 ([6]) *The necessary conditions for the existence of a* URD$(\{3, 4\}; v)$ *with* $r_3, r_4 > 0$ *are:*

- $v \equiv 0$ (mod12);

- *r_4 is odd;*
- *if $r_k > 1$, then $v \geq k^2$; and*
- $r_4 = \frac{v-1-2\,r_3}{3}$ $\left(r_3 = \frac{v-1-3\,r_4}{2}\right)$.

The fourth condition means that if r_3 is given, then r_4 is determined, and vice versa.

Remark $r_3 \equiv 1 \pmod 3$.

Proof Because r_4 is odd, insert $2\,i+1$ for r_4 in the last equation of Theorem 1.9; this gives $r_3 = \frac{v}{2} - 3\,i - 2 \equiv -2 \equiv 1 \pmod 3$. □

Now some known results are summarized of URDs with block sizes 3 and 4. The next two theorems are special cases of Theorem 1.4. We take the groups as an additional parallel class to get the URDs.

Theorem 1.10 ([26]) *There exist an* RGDD(3, 4; v) *and also a* URD({3, 4}; v) *with* $r_4 = 1$ *if, and only if,* $v \equiv 0 \pmod{12}$.

Theorem 1.11 ([22,24,28,30]) *There exist an* RGDD(4, 3; v) *and so a* URD({3, 4}; v) *with* $r_3 = 1$ *if, and only if,* $v \equiv 0 \pmod{12}$, $v \geq 24$.

Danziger showed in [5]:

Theorem 1.12 ([5]) *There exists a* URD({3, 4}; v) *with* $r_4 = 3$ *for all*
$v \equiv 12 \pmod{24}$, $v \neq 12$ *with the possible exceptions of* $v = 84,\ 156$.

In [25] the author showed:

Theorem 1.13 [25] *There exists a* URD({3, 4}; v) *with* $r_4 = 3$ *or* 5 *if, and only if,* $v \equiv 0 \pmod{12}$, *except when* $v = 12$.

There is also a result for $K = \{3,\ 5\}$:

Theorem 1.14 ([26]) *There exists a* URD ({3, 5} ; v) *with* $r_5 = 2, 3, 4, 5$ *if, and only if,* $v \equiv 15 \pmod{30}$ *except* $v = 15$, *and except possibly*
$v = 105$ *for* $r_5 = 3$,
$v \in \{105, 165, 285, 345\}$ *for* $r_5 = 2, 4, 5$.

In the next section, labeled resolvable designs are introduced and designs are constructed for four exceptions in Theorem 1.14. Ingredient designs for recursive constructions, which are described in Sect. 3, are created by some new labeled uniformly resolvable designs. In the further sections the results are described.

2 Labeled resolvable designs and direct constructions

The concept of labeled resolvable designs is needed in order to get direct constructions for resolvable designs. This concept was introduced by Shen [27,29,30].

Let $(X,\ \boldsymbol{B})$ be a (U)GDD$_\lambda(K,\ M;\ v)$ where $X = \{a_1,\ a_2, \ldots, a_v\}$ is totally ordered with ordering $a_1 < a_2 < \cdots < a_v$. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, it is supposed that $x_1 < x_2 < \cdots < x_k$. Let Z_λ be the group of residues modulo λ .

Let $\varphi : \boldsymbol{B} \to Z_\lambda^{\binom{k}{2}}$ be a mapping where for each $B = \{x_1, x_2, \ldots, x_k\} \in \boldsymbol{B}, k \in K, \varphi(B) = (\varphi(x_1, x_2), \ldots, \varphi(x_1, x_k), \varphi(x_2, x_3), \ldots, \varphi(x_2, x_k), \varphi(x_3, x_4), \ldots, \varphi(x_{k-1}, x_k))$, $\varphi(x_i, x_j) \in Z_\lambda$for $1 \le i < j \le k$.

A (U)GDD$_\lambda(K,\ M;\ v)$ is said to be a *labeled (uniform resolvable) group divisible design*, denoted L(U)GDD$_\lambda(K,\ M;\ v)$, if there exists a mapping φ such that:

1. For each pair $\{x,\ y\} \subset X$ with $x < y$, contained in the blocks $B_1, B_2, \ldots, B_\lambda$, then $\varphi_i(x, y) \equiv \varphi_j(x, y)$ if, and only if, $i = j$ where the subscripts i and j denote the blocks to which the pair belongs, for $1 \le i, j \le \lambda$; and
2. For each block $B = \{x_1, x_2, \ldots, x_k\}, k \in K, \varphi(x_r, x_s) + \varphi(x_s, x_t) \equiv \varphi(x_r, x_t) \pmod{\lambda}$, for $1 \le r < s < t \le k$.

The blocks will be denoted in the following form:

$$(x_1x_2 \ldots x_k;\ \varphi(x_1, x_2) \ldots \varphi(x_1, x_k)\varphi(x_2, x_3) \ldots \varphi(x_2, x_k)\varphi(x_3, x_4) \ldots \varphi(x_{k-1}, x_k)),\ k \in K.$$

The above definition was first given in [25] and is a little bit more general than the definition by Shen [30] with $K = \{k\}$ or Shen and Wang [29] for transversal designs. A special case of type 1^v, a labeled URD$_\lambda(K;\ v)$, is denoted by LURD$_\lambda(K;\ v)$. A labeled K -frame of type T and index λ is denoted K-LF$_\lambda$ of type T.

The main application of the labeled designs is to blow up the point set of a given design with the following theorem (Shen, [17]) here extended for labeled (uniform resolvable) pairwise balanced designs.

Theorem 2.1 ([17,25]) *If there exists an* L(U)GDD$_\lambda(K,\ M;\ v)$ *(with r_k^L classes of size k, for each $k \in K$), then there exists a* (U)GDD$(K,\ \lambda M;\ \lambda v)$*, where $\lambda M = \{\lambda g_i | g_i \in M\}$ (with $r_k = r_k^L$ classes of size k, for each $k \in K$). If there exists a uniform frame $K -$* LF$_\lambda$ *of type T, then there exists a uniform K-*frame *of type λT, where $\lambda T = \{\lambda g_i | g_i \in T\}$.*

Proof Let $(X,\ \boldsymbol{G},\ \boldsymbol{B})$ be an LRGDD$_\lambda(K,\ M;\ v)$ where $X = \{a_1,\ a_2, \ldots, a_v\}$. Expanding each point $a_i \in X$ λ times gives the points $\{a_{i,0}, \ldots, a_{i,\lambda-1}\}, i = 1, \ldots, v$, in the new design. Any group with g_i points becomes a new group with $\lambda \cdot g_i$ points. Each labeled block, $(x_1x_2 \ldots x_k;\ \varphi(x_1, x_2) \ldots \varphi(x_1, x_k)\varphi(x_2, x_3) \ldots \varphi(x_2, x_k)\varphi(x_3, x_4) \ldots \varphi(x_{k-1}, x_k))$, $k \in K$, gives λ new blocks $\{x_{1,j}, x_{2,j+\varphi(x_1,x_2)}, \ldots, x_{k,j+\varphi(x_1,x_k)}\}, k \in K, j = 0, \ldots, (\lambda - 1)$ with indices calculated mod(λ) and all blocks taken together consist of different points. Therefore, each (partial) uniform parallel class of the labeled design with blocks of size k gives a (partial) parallel class of the expanded design with blocks of the same size k. For each pair $\{x,\ y\} \subset X$ with $x < y$ from different groups, let $B_1,\ B_2, \ldots, B_\lambda$ be the λ blocks containing $\{x,\ y\}$ and let $\varphi_i(x,\ y)$ be the values of $\varphi(x,\ y)$ corresponding to $B_i,\ 1 \le i \le \lambda$. Due to the first condition on φ , all pairs $\{x_j, y_{j+\varphi_i(x,y)}\}, i = 1, \ldots, \lambda, j = 0, \ldots, (\lambda - 1)$, are different, where the indices are calculated mod(λ). □

A special case for URDs is shown in the following.

Corollary 2.2 *If there exists an* LURD$_\lambda(K;\ v)$ *with r_k^L classes of size k, for each $k \in K$, then there exists a* URD$(K \cup \{\lambda\};\ \lambda v)$ *with $r_k = r_k^L$ when $k \ne \lambda$, and $r_\lambda = r_\lambda^L + 1$, where we take $r_\lambda^L = 0$ if $\lambda \notin K$.*

Lemma 2.3 *There exist an* LURD$_3(\{3, 4\};\ 12)$ *with $r_4 = 7$, an* LURD$_4(\{3, 4\};\ 12)$ *with $r_4 = 6$, an* LURD$_4(\{3, 4\};\ 12)$ *with $r_4 = 8$ and an* LURD$_4(\{3, 4\};\ 12)$ *with $r_3 = 7$.*

Proof The desired designs were found computationally.

An $LURD_3(\{3,4\};\ 12)$ with $r_4 = 7$; each row forms a parallel class:

```
(7 11 12; 1 0 2), (1  2 10; 1 1 0), (4  8  9; 0 2 2), (3  5  6; 1 1 0),
(1  6  7; 2 1 2), (8  9 11; 0 0 0), (4  5 10; 2 0 1), (2  3 12; 2 2 0),
(6  9 10; 0 1 1), (4  8 12; 2 1 2), (1  5  7; 0 2 2), (2  3 11; 0 1 1),
(3  6  9; 0 1 1), (4 11 12; 2 2 0), (2  8 10; 0 1 1), (1  5  7; 2 0 1),
(1  3  9; 1 0 2), (5  8 11; 0 1 1), (2  4 10; 0 2 2), (6  7 12; 0 2 2),
(1  3  8; 0 2 2), (5  9 12; 2 1 2), (4  7 10; 0 1 1), (2  6 11; 0 2 2),
(1  9 10 11; 2 2 1 0 2 2), (2  5  6  8; 1 2 2 1 1 0), (3  4  7 12; 2 1 2 2 0 1),
(1  9 10 12; 1 0 2 2 1 2), (2  5  7  8; 2 2 1 0 2 2), (3  4  6 11; 1 2 2 1 1 0),
(2  3  7  9; 1 1 1 0 0 0), (6  8 10 12; 2 2 0 0 1 1), (1  4  5 11; 0 1 0 1 0 2),
(3  5 10 11; 0 0 0 0 0 0), (2  4  7  9; 2 0 2 1 0 2), (1  6  8 12; 0 1 1 1 1 0),
(2  5  9 12; 0 0 0 0 0 0), (1  3  4  8; 2 2 0 0 1 1), (6  7 10 11; 1 0 1 2 0 1),
(1  2 11 12; 2 2 0 0 1 1), (3  7  8 10; 2 0 2 1 0 2), (4  5  6  9; 0 2 1 2 1 2),
(7  8  9 11; 0 1 2 1 2 1), (1  2  4  6; 0 1 1 1 1 0), (3  5 10 12; 2 1 1 2 2 0).
```

An $LURD_4(\{3,4\};\ 12)$ with $r_4 = 6$; each row forms a parallel class:

```
(2  3  9; 3 2 3), (1  4  6; 1 1 0), (8 11 12; 1 1 0), (5  7 10; 3 3 0),
(6 11 12; 0 2 2), (4  7  8; 2 2 0), (3  5  9; 2 2 0), (1  2 10; 0 1 1),
(1  5 12; 0 2 2), (2  4  9; 3 1 2), (8 10 11; 1 2 1), (3  6  7; 1 2 1),
(2  4  8; 1 1 0), (1  3 10; 1 0 3), (5  6 12; 1 1 0), (7  9 11; 2 1 3),
(5  9 11; 3 0 1), (2  3  8; 1 2 1), (1  4  7; 0 1 1), (6 10 12; 1 1 0),
(5  7  8; 1 0 3), (1  2 11; 2 3 1), (3 10 12; 0 2 2), (4  6  9; 2 3 1),
(3  5 12; 3 3 0), (2  6 11; 3 2 3), (1  7  8; 3 0 1), (4  9 10; 1 2 1),
(7  9 12; 0 1 1), (2  5 11; 1 0 3), (6  8 10; 3 3 0), (1  3  4; 0 3 3),
(1  9 12; 3 1 2), (4  5 11; 0 1 1), (2  6  7; 0 0 0), (3  8 10; 3 1 2),
(1  8 11; 1 1 0), (3  6  9; 0 0 0), (2  5  7; 2 2 0), (4 10 12; 3 0 1),
(6  7  8; 2 0 2), (2  9 10; 3 2 3), (1  3 11; 3 0 1), (4  5 12; 3 2 3),
(5  6  9; 3 2 3), (1  2 10; 3 2 3), (3  7 11; 0 2 2), (4  8 12; 3 3 0),
(2  5 10; 3 0 1), (3  6  8; 2 0 2), (4 11 12; 2 1 3), (1  7  9; 2 1 3),
(1  2  7 12; 1 0 3 3 2 3), (4  6 10 11; 3 1 0 2 1 3), (3  5  8  9; 0 2 1 2 1 3),
(1  8  9 12; 2 0 0 2 2 0), (5  7 10 11; 2 0 2 2 0 2), (2  3  4  6; 2 0 1 2 3 1),
(1  3  4  5; 2 2 3 0 1 1), (8  9 10 11; 1 3 3 2 2 0), (2  6  7 12; 2 1 1 3 3 0),
(2  3 11 12; 0 3 0 3 0 1), (1  5  6  8; 2 2 3 0 1 1), (4  7  9 10; 3 0 0 1 1 0),
(2  4  5  8; 2 0 3 2 1 3), (1  6  9 11; 0 2 2 2 2 0), (3  7 10 12; 3 2 1 3 2 3),
(3  4  7 11; 1 1 0 0 3 3), (2  8  9 12; 0 0 3 0 3 3), (1  5  6 10; 1 3 3 2 2 0).
```

An $LURD_4(\{3,4\};12)$ with $r_4 = 8$; each row forms a parallel class:

```
(1  2  6; 2 3 1), (3  7 10; 1 2 1), (4  8  9; 1 1 0), (5 11 12; 0 1 1),
(1  4  6; 0 0 0), (2  3  8; 1 2 1), (5  7 12; 3 0 1), (9 10 11; 1 0 3),
(1  2  3; 1 1 0), (4  5 11; 0 3 3), (6  8 12; 1 0 3), (7  9 10; 2 0 2),
(1  6  9; 1 2 1), (3  5 10; 3 3 0), (7  8 11; 3 2 3), (2  4 12; 2 2 0),
(2 10 11; 0 0 0), (3  6 12; 1 2 1), (1  7  8; 1 2 1), (4  5  9; 2 2 0),
(2  7  9; 3 3 0), (1  8 11; 0 1 1), (3  4 12; 3 0 1), (5  6 10; 0 1 1),
(1  4  8; 2 1 3), (3  9 12; 3 1 2), (2  5 10; 3 1 2), (6  7 11; 0 1 1),
(1  4  9; 3 3 0), (2  5  8; 0 0 0), (6 10 12; 3 2 3), (3  7 11; 2 2 0),
(2  4 12; 0 3 3), (1  7 10; 2 1 3), (3  6 11; 0 3 3), (5  8  9; 1 3 2),
(2  3  8; 3 1 2), (1  5  7; 3 0 1), (4  6 11; 2 0 2), (9 10 12; 3 0 1),
(2  5  6  8; 1 3 3 2 2 0), (1  7  9 12; 3 0 1 1 2 1), (3  4 10 11; 0 0 1 0 1 1),
(1  2 11 12; 3 0 3 1 0 3), (4  7  8 10; 0 2 2 2 2 0), (3  5  6  9; 0 3 1 3 1 2),
(4  6  7 12; 3 2 2 3 3 0), (1  2  3 10; 0 2 3 2 3 1), (5  8  9 11; 3 2 1 3 2 3),
(3  8  9 12; 3 0 3 1 0 3), (1  5  6 10; 1 2 0 1 3 2), (2  4  7 11; 1 0 3 3 2 3),
(1  5 11 12; 0 2 2 2 2 0), (2  4  9 10; 3 2 2 3 3 0), (3  6  7  8; 2 0 0 2 2 0),
(2  5  7 12; 2 2 1 0 3 3), (1  3  9 11; 3 1 3 2 0 2), (4  6  8 10; 1 0 1 3 0 1),
(2  6  7  9; 0 1 0 1 0 3), (1  3  4  5; 0 1 2 1 2 1), (8 10 11 12; 2 0 2 2 0 2),
(2  6  9 11; 2 1 2 3 0 1), (3  4  5  7; 2 1 3 3 1 2), (1  8 10 12; 3 2 0 3 1 2).
```

An $LURD_4(\{3,4\};\ 12)$ with $r_3 = 7$; each row forms a parallel class:

```
(3 10 11; 0 1 1), (1  4  8; 3 1 2), (7  9 12; 3 0 1), (2  5  6; 1 1 0),
(3  5  6; 1 3 2), (1  7 12; 2 0 2), (4  8 11; 0 1 1), (2  9 10; 3 0 1),
(2  5 12; 0 3 3), (8  9 10; 3 1 2), (1  3  7; 3 0 1), (4  6 11; 3 0 1),
(8 10 12; 3 3 0), (5  6  7; 3 2 3), (2  4  9; 1 1 0), (1  3 11; 2 1 3),
(1  4 11; 0 2 2), (3  7  9; 2 3 1), (2  6 12; 0 2 2), (5  8 10; 0 0 0),
(7 10 11; 0 3 3), (1  5  9; 0 1 1), (2  4 12; 0 1 1), (3  6  8; 0 2 2),
(1  6  9; 3 2 3), (2  3  7; 3 3 0), (4  5  8; 3 1 2), (10 11 12; 2 3 1),
(4  6  9 10; 0 1 1 1 1 0), (2  7  8 11; 0 0 0 0 0 0), (1  3  5 12; 0 2 3 2 3 1),
(3  8 10 12; 0 2 0 2 0 2), (1  6  9 11; 1 3 0 2 3 1), (2  4  5  7; 2 2 1 0 3 3),
(6  7 11 12; 2 0 3 2 1 3), (1  2  4 10; 3 2 0 3 1 2), (3  5  8  9; 0 3 0 3 0 1),
(5  6  8 11; 1 1 3 0 2 2), (3  4  7  9; 2 3 1 1 3 2), (1  2 10 12; 2 1 2 3 0 1),
(7  8  9 12; 2 0 3 2 1 3), (3  4  6 10; 0 1 3 1 3 2), (1  2  5 11; 0 3 3 3 3 0),
(4  6  9 12; 2 2 2 0 0 0), (5  7 10 11; 0 1 1 1 1 0), (1  2  3  8; 1 1 2 0 1 1),
(1  5  9 10; 1 0 3 3 2 3), (2  6  7  8; 2 2 3 0 1 1), (3  4 11 12; 3 2 2 3 3 0),
(1  6  8 12; 0 3 1 3 1 2), (4  5  7 10; 1 2 0 1 3 2), (2  3  9 11; 2 0 2 2 0 2),
(1  6  7 10; 2 3 2 1 0 3), (2  8  9 11; 2 2 1 0 3 3), (3  4  5 12; 1 3 1 2 0 2),
(1  4  7  8; 1 1 0 0 3 3), (2  3  6 10; 1 3 2 2 1 3), (5  9 11 12; 2 2 0 0 2 2).
```

□

Lemma 2.4 *There exist a* $URD(\{3,4\};\ 36)$ *with* $r_4 = 7$, *a* $URD(\{3,4\};\ 48)$ *with* $r_4 = 7$, *a* $URD(\{3,4\};\ 48)$ *with* $r_3 = 10$, $r_4 = 9$, *a* $URD(\{3,4\};\ 48)$ *with* $r_3 = 7$, *a* $URD(\{3,4\};\ 36)$ *with* $r_4 = 9$, *a* $URD(\{3,4\};\ 60)$ *with* $r_4 = 7$ *or* 9, *a* $URD(\{3,4\};\ 72)$ *with* $r_4 = 9$, *a* URD($\{3,4\}$; 132) *with* $r_4 = 7$ *or* 9*, a* URD($\{3,4\}$; 156) *with* $r_4 = 7$ *or* 9*, a* URD($\{3,4\}$; 204) *with* $r_4 = 7$ *or* 9 *and a* URD($\{3,4\}$; 228) *with* $r_4 = 9$.

Proof For the first four URDs the assertion follows with Corollary 2.2 and Lemma 2.3. For the next case we begin with a well-known RTD(4, 9). Filling the groups with RPBD(3; 9) results in a URD($\{3,4\}$; 36) with $r_4 = 9$. All other URDs are given in the Online Resource. □

Four possible exceptions of Theorem 1.14 are constructed in the next theorem.

Theorem 2.5 *There exists a* URD ($\{3, 5\}$; v) *with* $r_5 = 2, 3, 4, 5$ *if, and only if,* $v \equiv 15 \pmod{30}$ *except* $v = 15$*, and except possibly* $v \in \{165, 285, 345\}$ *for* $r_5 = 2, 4, 5$.

Proof By Theorem 1.14 only URDs with $v = 105$ and $r_5 = 2, 3, 4$ or 5 are needed. A uniform $\{3,5\}$-LRGDD$_7$ of type 3^5 with $r_5 = 2, 3, 4$ or 5 is given in the Online Resource, therefore, by Theorem 2.1 there exists a uniform $\{3,5\}$-RGDD of type 21^5 with $r_5 = 2$, 3, 4 or 5. By filling all groups with a RPBD(3; 21), we obtain a URD ($\{3, 5\}$; 105) with $r_5 = 2, 3, 4$ or 5. □

3 Recursive constructions

We now describe some constructions which we will use later. Filling groups and holes with PBDs or GDDs are known basic constructions [8,11]. Here groups and holes are filled with URDs to get new URDs.

Construction 3.1 (Filling in groups) *Suppose there exists an* RGDD(k_1, g; ig) *and a* URD ($\{k_1, k_2\}$; g) *with* $r_{k_2} = j$*, then there exists a* URD ($\{k_1, k_2\}$; ig) *with* $r_{k_2} = j$ *and an* IURD ($\{k_1, k_2\}$; ig) *with a hole of size g,* $r_{k_1} = \frac{(i-1)g}{k_1-1}$ k_1 *-pcs,* $r^{\circ}_{k_1} = \frac{g-1-(k_2-1)j}{k_1-1}$ *holey (or partial)* k_1 *-pcs,* $r_{k_2} = 0$ k_2 *-pcs and* $r^{\circ}_{k_2} = j$ *holey* k_2 *-pcs.*

Proof Fill all groups of the RGDD with the URD to obtain the URD. Leave exactly one group empty to get the IURD. □

Construction 3.2 (Generalized frame construction) *Suppose there is a* k_1 *-frame of type* $T = \{t_i : i = 1, \ldots, n\}$*. Let* $v = \sum_{i=1}^{n} t_i$ *. If, for each* $i = 1, \ldots, n$*, there exists an* IURD ($\{k_1, k_2\}$; $t_i + s$) *with a hole of size s,* $r_{k_1} = \frac{t_i}{k_1-1}$, $r^{\circ}_{k_1} = \frac{s-1-(k_2-1)j_2}{k_1-1}$, $r_{k_2} = 0$ *and* $r^{\circ}_{k_2} = j_2$ *for* $i = 1, \ldots, n-1$*, then there exists an* IURD ($\{k_1, k_2\}$; $v + s$) *with a hole of size* $t_n + s$*,* $r_{k_1} = \frac{\sum_{i=1}^{n-1} t_i}{k_1-1}$, $r^{\circ}_{k_1} = \frac{t_n}{k_1-1} + \frac{s-1-(k_2-1)j_2}{k_1-1}$, $r_{k_2} = 0$ *and* $r^{\circ}_{k_2} = j_2$ *. If there exists a* URD ($\{k_1, k_2\}$; $t_n + s$) *with* $r_{k_2} = j_2$ *and therefore* $r_{k_1} = \frac{t_n}{k_1-1} + \frac{s-1-(k_2-1)j_2}{k_1-1}$*, then a* URD ($\{k_1, k_2\}$; $v + s$) *with* $r_{k_2} = j_2$ *exists.*

Proof Let X be the point set of the frame, we construct the new design on $X \cup S$, where S is a set of s new points which will be the hole. For each group T_i of size t_i, $i = 1, \ldots, n-1$, of the frame we fill $T_i \cup S$ with an URD ($\{k_1, k_2\}$; $t_i + s$) so that the hole covers S. This gives the IURD. Each group of the frame with size t_i has $\frac{t_i}{k_1-1}$ k_1 -pcs, which can be extended with the k_1 -pcs from the IURD. The holey pcs from all the IURDs combine to form holey pcs of the new IURD ($\{k_1, k_2\}$; $v + s$) with a hole of size $t_n + s$. These give j_2 holey k_2 -pcs and $\frac{s-1-(k_2-1)j_2}{k_1-1}$ holey k_1 -pcs. $\frac{t_n}{k_1-1}$ holey k_1 -pcs are from the group of size t_n of the frame. Filling the last hole with the URD ($\{k_1, k_2\}$; $t_n + s$) with $r_{k_2} = j_2$ results in the URD ($\{k_1, k_2\}$; $v + s$) with $r_{k_2} = j_2$. □

Remark If in Construction 3.2 all IURDs come from Construction 3.1 with $r^{\circ}_{k_2} = j_2$ and the URD has $r_{k_2} = j_2$, then all additional conditions in Construction 3.2 are fulfilled. In this paper all IURDs come from Construction 3.1.

Construction 3.3 *(Weighting) [5] Let $(X, \mathbf{G}, \mathbf{B})$ be a GDD, and let $w : X \to Z^+ \cup 0$ be a weight function on X. Suppose that for each block $B \in \mathbf{B}$, there exists a k-frame of type $\{w(x) : x \in B\}$. Then there is a k-frame of type $\left\{ \sum_{x \in G_i} w(x) : G_i \in \mathbf{G} \right\}$.*

The next two lemmas are similar to a lemma in [17], but here the lemmas are applied to URDs.

Lemma 3.4 *Let $u \subset \{0, 1\}$, $s \equiv 0 \pmod{12}$, $s \geq 12$. Suppose a $\mathrm{TD}(6 + u, m)$ exists. Suppose also that there exists a $\mathrm{URD}(\{k, 4\};\ 12 \cdot b + s)$ with $r_k = j$, where $0 \leq b \leq m$, an $\mathrm{IURD}(\{k, 4\};\ 12m + s)$ with $r^{\circ}_k = j$ and a hole of size s and an $\mathrm{IURD}(\{k, 4\};\ 12a + s)$ with $r^{\circ}_k = j$ and a hole of size s, when $u = 1$, where $0 \leq a \leq m$, which all fulfil the conditions of Construction 3.2. Then there exists a $\mathrm{URD}(\{k, 4\};\ 12i)$ with $r_k = j$, where $i = 5m + ua + b + \frac{s}{12}$.*

Proof Truncate a group in the $\mathrm{TD}(6 + u, m)$ to size b. For $u = 1$ truncate another group to size a. This gives a $\mathrm{GDD}(\{5, 6, 6 + u\}, \{m, b, au\};\ v)$. Apply Construction 3.3 with weight 12 and 4-frames of types 12^t for $t \in \{5, 6, 6 + u\}$, which exist by Theorem 1.6. The result is a 4-frame of type $(12m)^5(12b)^1(12au)^u$. Adjoin s infinite points and apply Construction 3.2 with the above URD and IURDs, which gives the design as required. □

Lemma 3.5 *Let $s \equiv 0 \pmod{12}$, $s \geq 12$, $u \subset \{0, 1\}$. Suppose a $\mathrm{GDD}(\{4, 5, 5 + u\}, \{m, b, au\};\ v)$ exists. Suppose also that there exist a $\mathrm{URD}(\{3,k\};\ 12b + s)$ with $r_k = j$, where $0 \leq b \leq m$, an $\mathrm{IURD}(\{3, k\};\ 12m + s)$ with $r^{\circ}_k = j$ and a hole of size s, and an $\mathrm{IURD}(\{3, k\};\ 12a + s)$ with $r^{\circ}_k = j$ and a hole of size s, when $u = 1$, where $0 \leq a \leq m$, which all fulfil the conditions of Construction 3.2. Then there exists a $\mathrm{URD}(\{3, k\};\ 12i)$ with $r_k = j$, where $i = 4m + ua + b + \frac{s}{12}$.*

Proof Apply Construction 3.3 with weight 12 and 3-frames of types 12^t for $t \in \{4, 5, 5+u\}$, which exist by Theorem 1.5. The result is a 3-frame of type $(12m)^4(12b)^1(12au)^u$. Adjoin s infinite points and apply Construction 3.2 with the above URD and IURDs, which gives the design as required. □

Lemma 3.6 *If there exists a $\mathrm{URD}(\{3, 4\};\ v_0)$ with $r_4 = m > 0$, then there exist a $\mathrm{URD}(\{3, 4\};\ nv_0)$ with $r_4 = m$ and an $\mathrm{IURD}(\{3, 4\};\ nv_0)$ with $r^{\circ}_4 = m$ and a hole of size v_0 for all $n \geq 3$.*

Proof Since there exists a $\mathrm{URD}(\{3, 4\};\ v_0)$ with $r_4 = m > 0$, we have $v_0 \equiv 0 \pmod{12}$ by Theorem 1.7. The assertion follows from Theorem 1.4 by filling a 3-RGDD of type v_0^n by the $\mathrm{URD}(\{3, 4\};\ v_0)$ with $r_4 = m$. By not filling one group, we get the IURD. □

Some further needed constructions from Danziger [5].

Lemma 3.7 ([5], Theorem 2.5) *If there exists a uniform $\{3, 4\}$ -frame of type $(g_1;\ 3^{\frac{g_1}{2}})^t(g_2;\ 3^{\frac{g_2-3r}{2}}\ 4^r)^1$ and $w \equiv 3 \pmod{6}$ is such that $g_1 + w \equiv 3 \pmod{6}$, $2w \leq g_1$, and there exists a $\mathrm{URD}(\{3, 4\}; g_2 + w)$ with $r_4 = r$, $\left(r_3 = \frac{g_2+w-1-3r}{2}\right)$, then there exists a $\mathrm{URD}(\{3, 4\}; g_1t + g_2 + w)$ with $r_4 = r$.*

Lemma 3.8 ([5], Lemmas 3.3 and 3.4) *Let* $v_0 \equiv 0 \pmod{12}$, r_4 *odd.*

For $v_0 = 9 \cdot r_4 + 6j + 9$ *with* j *and integer,* $j \geq 0$, *there exists a uniform* $\{3, 4\}$ *-frame of type* $(24;\ 3^{12})^t (v_0 - 9;\ 3^{3(r_4+j)}\ 4^{r_4})^1$ *for all* $t \equiv 1 \pmod 3$ *with* $t \geq 1 + \frac{3(r_4+j)}{4}$.

For $v_0 = 9r_4 + 6j + 3$ *with* j *and integer,* $j \geq 0$, *there exists a uniform* $\{3, 4\}$ *-frame of type* $(24;\ 3^{12})^t (v_0 - 3;\ 3^{3(r_4+j+1)}\ 4^{r_4})^1$ *for all* $t \equiv 1 \pmod 3$ *with* $t \geq 1 + \frac{3(r_4+j)}{4}$.

Lemma 3.8 is a little bit more general then the lemmas by Danziger [5], but the proof is analogues. The second variant of Lemma 3.8 is only useful if $j = 0$. In all other cases the first variant is more effective, because the bound for t is lesser.

4 Results for URDs with exactly 4 parallel classes with blocks of size 3

Theorem 4.1 *There exists a* URD$(\{3, 4\};\ 12i)$ *with* $r_3 = 4$ *if, and only if,* $i \geq 1$ *integer.*

Proof There exists a URD$(\{3, 4\};\ 12)$ with $r_3 = 4$ and $r_4 = 1$ by Theorem 1.10, a URD$(\{3, 4\};\ 24)$ with $r_3 = 4$ and $r_4 = 5$ by Theorem 1.12, and a URD$(\{3, 4\};\ 36)$ with $r_3 = 4$ and $r_4 = 9$ by Lemma 2.4. Because of Theorem 1.4 all 4-RGDDs of type 12^n exist for $n \geq 4$. Filling the groups with the URD$(\{3, 4\};\ 12)$ with $r_3 = 4$, gives the desired designs. □

5 Results for URDs with exactly 7 parallel classes with blocks of size 3

Now the aim is to find {3,4}-URDs with $r_3 = 7$. In this section let $N = \{n : \exists\ \text{URD}(\{3, 4\};\ 12 \cdot n) \text{with } r_3 = 7\}$.

Lemma 5.1 *There exist a* URD$(\{3, 4\};\ 24)$, *a* URD$(\{3, 4\};\ 36)$ *and a* URD$(\{3, 4\};\ 48)$ *all with* $r_3 = 7$.

Proof A URD$(\{3, 4\};\ 24)$ with $r_3 = 7$ and $r_4 = 3$ exists by Theorem 1.13. A URD$(\{3, 4\};\ 36)$ with $r_3 = 7$ and $r_4 = 7$ exists by Lemma 2.4. A URD$(\{3, 4\};\ 48)$ with $r_3 = 7$ and $r_4 = 11$ exists by Lemma 2.4. □

Lemma 5.2 *There exists a* URD$(\{3, 4\};\ 24\ i)$ *with* $r_3 = 7$ *and an* IURD$(\{3, 4\};\ 24\ i)$ *with* $r_3^\circ = 7$, $r_4 = 8\ (i - 1)$, $r_4^\circ = 3$ *and a hole of size* 24 *for* $i \geq 4$.

Proof Let $g = 24$ in Construction 3.1. The 4-RGDDs of type 24^i exist by Theorem 1.4. A URD$(\{3, 4\};\ 24)$ with $r_3 = 7$, $r_4 = 3$ exists by Lemma 5.1. □

Lemma 5.3 *There exists a* URD({3,4}; 60i) *with* $r_3 = 7$ *and an* IURD $(\{3, 4\}; 60i)$ *with* $r_3^\circ = 7$ *and a hole of size* 15 *for* $i \geq 1$.

Proof By Theorem 1.4 there exists a 4-RGDD of type $15^{4 \cdot i}$ for $i \geq 1$. Fill the groups with the well known RPBD(3; 15), which has seven parallel classes. By not filling one group we obtain IURDs. □

Lemma 5.4 *There exists a* URD$(\{3, 4\};\ 36\ i)$ *with* $r_3 = 7$ *for* $i \geq 4$ *and an* IURD$(\{3, 4\};\ 36\ i)$ *with* $r_3^\circ = 7$, $r_4^\circ = 7$ *and a hole of size* 36 *for* $i \geq 4$, $i \notin \{14, 15\}$.

Proof Let $g = 36$ in Construction 3.1, by Theorem 1.4 RGDDs of type 36^i exist for $i \geq 4, i \notin \{11, 14, 15, 18, 23\}$. A URD$(\{3,4\};\ 36)$ with $r_3 = 7$ exists by Lemma 2.4. By not filling in one group we obtain the IURDs. For $i = 14$ and $i = 18$ the URDs exist by Lemma 5.2 and for $i = 15$ by Lemma 5.3. For $i = 18$ the IURD exists by Lemma 5.13. For $i = 23$ a URD and also an IURD with a hole of size 36 is constructed in Lemma 5.11. Now the proof for the last case $i = 11$. There exists a 5-GDD of type $24^5 4^1$ by Theorem 1.2. Apply Construction 3.3 with weight 3 and 4-frames of type 3^5, which exist by Theorem 1.6. The result is a 4-frame of type $72^5 \cdot 12^1$. There exists an IURD$(\{3,4\};\ 96)$ with $r_3^\circ = 7$ and a hole of size 24 by Lemma 5.2. Adjoin 24 infinite points to the frame and fill all groups of size 72 with this IURD, with the infinite points forming the hole. The 24 frame 4-pcs are completed with the 24 complete 4-pcs of the IURD. This is the desired IURD. Fill the group of size 12 together with the infinite points with a URD$(\{3,4\};\ 36)$ with $r_3 = 7, r_4 = 7$. After completing the four frame 4-pcs it remain three more 4-pcs, which can be completed with the holey 4-pcs from the groups of size 72. □

Lemma 5.5 *Let* $0 \leq b \leq 3n$, $b \in \{0,1,2\} \cup \{5,7,9,\ldots\} \cup \{9,12,15,18,21,24,27\}$ *and* $n \geq 3$, $n \notin \{6,13,14\}$. *Then there exists a* URD$(\{3,4\};\ 12i)$ *with* $r_3 = 7$ *where* $i = 15n + b + 3$.

Proof The lemma is a special case of Lemma 3.4, let $s = 36$, $j = 7$ and $u = 0$. The URD$(\{3,4\};\ 12b + 36)$ with $r_3 = 7$ exists for $b \in \{0,1,2\} \cup \{5,7,9,...\} \cup \{9,12,15,18,21,24,27\}$ due to the Lemmas 5.1–5.4. Let $m = 3n$ then there exists an IURD$(\{3,4\};\ 12m + 36)$ with $r_3^\circ = 7$ and a hole of size 36 by Lemma 5.4 for $n + 1 \geq 4, n+1 \notin \{14,15\}$, that means $n \geq 3, n \notin \{13,14\}$. A TD$(6, 3n)$ exists for $3n \geq 5, 3n \neq 18$ by Theorem 1.3. Both conditions together give: $n \geq 3, n \notin \{6,13,14\}$. □

Lemma 5.6 *Let* $0 \leq b \leq 3n$, $b \in \{0,1,2\} \cup \{5,7,9,\ldots\} \cup \{9,12,15,18,21,24,27\}$, $3 \leq c \leq n$, $c \notin \{13,14\}$ *and* $n \geq 3, n \notin \{5,6,10,13,14,20\}$. *Then there exists a* URD$(\{3,4\};\ 12\ i)$ *with* $r_3 = 7$ *where* $i = 15n + b + 3c + 3$.

Proof The lemma is a special case of Lemma 3.4, let $s = 36$, $j = 7$ and $u = 1$. The condition on b is as in Lemma 5.5. Let $m = 3n$ then the IURD$(\{3,4\};\ 12m + 36)$ with $r_3^\circ = 7$ and a hole of size 36 exists by Lemma 5.4 for $n + 1 \geq 4, n + 1 \notin \{14,15\}$, that means $n \geq 3, n \notin \{13,14\}$. A TD$(7,\ 3n)$ exists for $3n \geq 7, 3n \notin \{15,18,30,60\}$ by Theorem 1.3. Both conditions together give $n \geq 3, n \notin \{5,6,10,13,14,20\}$. Let $a = 3c$ then it follows how above that there exists an IURD$(\{3,4\};\ 12a + 36)$ with $r_3^\circ = 7$ and a hole of size 36 for $c \geq 3, c \notin \{13,14\}$. □

With $b \in \{0,1,2\}$ and $c \in \{3,4,5,6,7\}$ all residue classes modulo 15 are covered:

Lemma 5.7 *Let* $b \in \{0,1,2\}$, $c \in \{3,4,5,6,7\}$ *and* $n \geq 3, n \notin \{5,6\}$. *Then there exists a* URD$(\{3,4\};\ 12\ i)$ *with* $r_3 = 7$, *where* $i = 15n + b + 3c + 3$, *that means* $\{i : i > 15 \cdot 6 + 2 + 3 \cdot 7 + 3 = 116\} \subset N$.

Proof For $n \notin \{10,13,14,20\}$ all the conditions of Lemma 5.6 are fulfilled. For $n \in \{10,13,20\}$ we have $i = 15n + b + 3c + 3 = \begin{cases} 15(n-1) + (b+12) + 3(c+1) + 3, & b \in \{0,\ 1\} \\ 15(n-1) + (b+15) + 3c + 3, & b = 2 \end{cases}$, $c \in \{3,4,5,6,7\}$ and for the right side of the above equation all the conditions of Lemma 5.6 are fulfilled. For $n=14$ we have $i=15n+b+3c+3=\begin{cases} 15(n-2)+(b+27)+3(c+1)+3, \\ \quad b\in\{0,\ 2\} \\ 15(n-1)+(b+24)+3(c+2)+3, \\ \quad b=1 \end{cases}$, $c \in \{3,4,5,6,7\}$ and for the right side of the above equation again all the conditions of Lemma 5.6 are fulfilled. □

Lemma 5.8 *Let* $0 \le b \le 2n$, $b \in \{0,1,2,3\} \cup \{6,8,10,\ldots\} \cup \{10,13,16,19\}$ *and* $n \ge 4$, $n \notin \{5,7,9,11\}$. *Then there exists a* URD$(\{3,4\};\ 12\ i)$ *with* $r_3 = 7$ *where* $i = 10{\cdot}n{+}b{+}2$.

Proof The lemma is a special case of Lemma 3.4, let $s = 24$, $j = 7$ and $u = 0$. A URD$(\{3,4\};\ 12\ b+24)$ with $r_3 = 7, r_4 = 4b+3$ exists for $b \in \{0,1,2,3\} \cup \{6,8,10,\ldots\} \cup \{10,13,16,19\}$ due to the Lemmas 5.1–5.4. Let $m = 2n$ then there exists an IURD$(\{3,4\};\ 12\ m+24)$ with $r_3^\circ = 7$ and a hole of size 24 by Lemma 5.2 for $n+1 \ge 4$, that means $n \ge 3$. Also the conditions of Construction 3.2 are fulfilled. A TD$(6,2n)$ exists for $2n \ge 5, 2n \notin \{6,10,14,18,22\}$ by Theorem 1.3. Both conditions together give $n \ge 4, n \notin \{5,7,9,11\}$. □

Lemma 5.9 *Let* $0 \le b \le 2n$, $0 \le$ c $\le n$, $b \in \{0,1,2,3\} \cup \{6,8,10,\ldots\} \cup \{10,13,16,19\}$, $c \ge 3$ *and* $n \ge 4, n \notin \{5,7,9,10,11,13,15,17,19,23,30\}$. *Then there exists a* URD$(\{3,4\};\ 12\ i)$ *with* $r_3 = 7$ *where* $i = 10 \cdot n + b + 2c + 2$.

Proof The lemma is a special case of Lemma 3.4, let $s = 24$, $j = 7$ and $u = 1$. The condition on b is as in Lemma 5.8. Let $m = 2n$ then it follows how above that an IURD$(\{3,4\};\ 12\ m+24)$ with $r_3^\circ = 7$ and a hole of size 24 exists for $n \ge 3$. A TD$(7,2n)$ exists for $2n \ge 7, 2n \notin \{10,14,18,20,22,26,30,34,38,46,60\}$ by Theorem 1.3. That gives $n \ge 4, n \notin \{5,7,9,10,11,13,15,17,19,23,30\}$. Let $a = 2c$ then it follows how above that an IURD$(\{3,4\};\ 12\ a+24)$ with $r_3^\circ = 7$ and a hole of size 24 exists for $c \ge 3$. □

Lemma 5.10 *We have* $\{43,\ 83\} \subset N$.

Proof In Lemma 5.8, where $i = 10n + b + 2$, let $n = 4$ or 8 and $b = 1$. □

Lemma 5.11 *We have* $\{49,53,69,71,73,77,89,91,97,101,103,107,109\} \subset N$.

Proof In Lemma 5.9, where $i = 10n + b + 2c + 2$, take the following values for n, b and c:

n	b	c	i
4	1	3	49
4	3	4	53
6	1	3	69
6	1	4	71
6	3	4	73
6	3	6	77
8	1	3	89
8	1	4	91
8	1	7	97
8	3	8	101
8	13	4	103
8	13	6	107
8	13	7	109

□

Lemma 5.12 *We have* $113 \in N$.

Proof In Lemma 5.5, where $i = 15n + b + 3$, let $n = 7$ and $b = 5$. □

Lemma 5.13 *We have* $\{59, 79\} \subset N$.

Proof In Lemma 5.6, where $i = 15n + b + 3c + 3$, take the following values for n, b and c:

n	b	c	i
3	2	3	59
4	7	3	79

□

Lemma 5.14 *We have* $\{61, 67\} \subset N$.

Proof There exists a TD$(8,\ m)$ for $m \in \{8,\ 9\}$ by [2]. Truncate one group in a TD$(8,\ m)$ to size b. This gives a GDD$(\{7,\ 8\},\ \{m,\ b\};\ v)$. Apply Construction 3.3 with weight 12 and 4-frames of types 12^t for $t \in \{7,\ 8\}$, which exist by Theorem 1.6. The result is a 4-frame of type $(12m)^7(12b)^1$.

Let $m = 8, b = 3$ and adjoin 24 infinite points and apply Construction 3.2 with an IURD$(\{3, 4\};\ 96 + 24)$ with $r_3^\circ = 7$ and a hole of size 24 from Lemma 5.2 and a URD$(\{3, 4\};\ 60)$ with $r_3 = 7$ from Lemma 5.3. The result is a URD$(\{3, 4\};\ 12 \cdot 61)$ with $r_3 = 7$.

Let $m = 9, b = 1$ and adjoin 36 infinite points and apply Construction 3.2 with an IURD$(\{3, 4\};\ 108^5 132^1)$ with $r_3^\circ = 7$ and a hole of size 36 from Lemma 5.4 and a URD$(\{3, 4\};\ 48)$ with $r_3 = 7$ from Lemma 5.1. The result is a URD$(\{3, 4\};\ 12 \cdot 67)$ with $r_3 = 7$. □

Lemma 5.15 *There exists a URD*$(\{3,4\}; 12\,n)$ *with* $r_3 = 7$ *for* $n = 41, 47$.

Proof There exists a TD(7, 7) by Theorem 1.3. Truncate a group of this design to size 3. Use one of the truncated point to redefine the groups. This gives a $\{6,\ 7\}$ -GDD of type $6^7 3^1$. Apply Construction 3.3 with weight 12 and 4-frames of types 12^6 and 12^7, which exist by Theorem 1.6. The result is a 4-frame of type $72^7 36^1$. There exists an IURD$(\{3, 4\};\ 96)$ with $r_3^\circ = 7$ and a hole of size 24 by Lemma 5.2. Adjoin 24 infinite points to the frame and fill all groups of size 72 with this IURD, with the infinite points forming the hole. Fill the group of size 36 together with the infinite points with a URD$(\{3,4\}; 60)$ with $r_3 = 7$. This gives a URD$(\{3, 4\};\ 12 \cdot 47)$ with $r_3 = 7$.

Start again from a TD(7, 7) and delete 6 points from a block to obtain a $\{6,\ 7\}$ -GDD of type $6^6 7^1$. Remove 4 points from the group of size 7 to get a $\{5, 6, 7\}$-GDD of type $6^6 3^1$. Apply Construction 3.3 with weight 12 and 4-frames of types 12^5, 12^6 and 12^7, which exist by Theorem 1.6. The result is a 4-frame of type $72^6 36^1$. Again, adjoin 24 infinite points to the frame and fill all groups of size 72 with this IURD, with the infinite points forming the hole. Fill the group of size 36 together with the infinite points with a URD$(\{3,4\}; 60)$ with $r_3 = 7$. This gives a URD$(\{3, 4\};\ 12 \cdot 41)$ with $r_3 = 7$. □

All lemmas of this section result in:

Theorem 5.16 *There exists a URD*$(\{3, 4\};\ 12\ n)$ *with* $r_3 = 7$ *if, and only if,* $n \geq 2$, *and possibly excepting the following 11 values:* $n \in \{6, 7, 9, 11, 13, 17, 19, 23, 29, 31, 37\}$.

6 Results for URDs with exactly 10 parallel classes with blocks of size 3

Now the aim is to find {3, 4}-URDs with $r_3 = 10$. In this section let $N = \{n : \exists\, \mathrm{URD}(\{3,4\};\ 12\cdot n) \text{with } r_3 = 10\}$.

Lemma 6.1 *There exist a* URD({3, 4}; 24)*, a* URD({3, 4}; 36) *and a* URD({3, 4}; 48) *all with* $r_3 = 10$.

Proof A design URD({3, 4}; 24) with $r_3 = 10$ and $r_4 = 1$ exists by Theorem 1.10. A URD({3, 4}; 36) with $r_3 = 10$ and $r_4 = 5$ exists by Theorem 1.13. A URD({3, 4}; 48) with $r_3 = 10$ and $r_4 = 9$ exists by Lemma 2.4. □

Lemma 6.2 *There exists a* URD({3, 4}; 84 i) *with* $r_3 = 10$ *for all integers* $i > 0$.

Proof There exists a 4-RGDD of type $21^{4\cdot i}$ for all integers i by Theorem 1.4. Fill the groups with the well known RPBD(3; 21), which has 10 parallel classes. □

Lemma 6.3 *There exists a* URD({3, 4}; 24 i) *with* $r_3 = 10$ *and an* IURD({3, 4}; 24 i) *with* $r_3^\circ = 10$ *and a hole of size* 24 *for* $i \geq 4$.

Proof A URD({3, 4}; 24) with $r_3 = 10$ exists by Theorem 1.10. Take in Construction 3.1 the value $g = 24$. The RGDDs exist by Theorem 1.4 for $i \geq 4$. By not filling one group we obtain the IURDs. □

Lemma 6.4 *There exists a* URD({3, 4}; 36 i) *with* $r_3 = 10$ *for* $i \geq 4$*,* $i \neq 15$ *and an* IURD({3, 4}; 36 i) *with* $r_3^\circ = 10$ *and a hole of size* 36 *for* $i \notin \{14, 15\}$.

Proof Take in Construction 3.1 the value $g = 36$. The RGDDs exist by Theorem 1.4 for $i \geq 4, i \notin \{11, 14, 15, 18, 23\}$. A URD({3, 4}; 36) exists by Lemma 6.1. For $i = 14$ the URD exists by Lemma 6.3. For $i = 18$ the URD and also the IURD are constructed in Lemma 6.13. For $i = 23$ the URD and also the IURD are constructed in Lemma 6.11. Now the proof for the last case $i = 11$. Take a 5-GDD of type $24^5 4^1$, which exists by Theorem 1.2. Apply Construction 3.3 with weight 3 and 4-frames of type 3^5, which exist by Theorem 1.6. The result is a 4-frame of type $72^5 12^1$. By Lemma 6.3 there exists an IURD({3, 4}; 96) with $r_3^\circ = 10$ and a hole of size 24. Adjoin 24 infinite points to the frame and fill all groups of size 72 with this IURD, with the infinite points forming the hole. The result is an IURD({3, 4}; $36 \cdot 11$) with $r_3^\circ = 10$ and a hole of size 36. Fill the hole with a URD({3, 4}; 36) with $r_3 = 10$, which is given in Lemma 6.1. □

Lemma 6.5 *Let* $0 \leq b \leq 2n$*,* $b \in \{0, 1, 2, 5\} \cup \{6, 8, 10, \ldots\} \cup \{10, 13, 16, 19\}$ *and* $n \geq 4, n \notin \{5, 7, 9, 11\}$. *Then there exists a* URD({3, 4}; 12 i) *with* $r_3 = 10$ *where* $i = 10n + b + 2$.

Proof The lemma is a special case of Lemma 3.4, let $s = 24$, $j = 10$ and $u = 0$. A URD({3, 4}; 12 b + 24) with $r_3 = 10$ exists for $b \in \{0, 1, 2, 5\} \cup \{6, 8, 10, \ldots\} \cup \{10, 13, 16, 19\}$ by the Lemmas 6.1–6.4. Let $m = 2n$ then an IURD({3, 4}; 12 m+24) with $r_3^\circ = 10$ and a hole of size 24 exists by Lemma 6.3 for $n + 1 \geq 4$, i.e. $n \geq 3$. A TD(6, 2n) exists for $2n \geq 5, 2n \notin \{6, 10, 14, 18, 22\}$ by Theorem 1.3. Both conditions together give $n \geq 4, n \notin \{5, 7, 9, 11\}$. □

Lemma 6.6 *Let* $0 \leq b \leq 2n$*,* $b \in \{0, 1, 2, 5\} \cup \{6, 8, 10, \ldots\} \cup \{10, 13, 16, 19\}$, $3 \leq c \leq n$ *and* $n \geq 4, n \notin \{5, 7, 9, 10, 11, 13, 15, 17, 19, 23, 30\}$. *Then there exists a* URD({3, 4}; 12 i) *with* $r_3 = 10$ *where* $i = 10n + b + 2c + 2$.

Proof The lemma is a special case of Lemma 3.4, let $s = 24$, $j = 10$ and $u = 1$. The condition on b is as in Lemma 6.5. Let $m = 2n$ then an IURD$(\{3, 4\}; 12m + 24)$ with $r_3^\circ = 10$ and a hole of size 24 exists by Lemma 6.3 for $n + 1 \geq 4$, i.e. $n \geq 3$. A TD$(7, 2n)$ exists for $2n \geq 7, 2n \notin \{10, 14, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$ by Theorem 1.3. Both conditions together give $n \geq 4, n \notin \{5, 7, 9, 10, 11, 13, 15, 17, 19, 23, 30\}$. Let $a = 2c$ then it follows as above that an IURD$(\{3, 4\}; 12a + 24)$ with $r_3^\circ = 10$ and a hole of size 24 exists for $c \geq 3$. □

Lemma 6.7 *Let* $0 \leq b \leq 3n$, $b \in \{0, 1, 4\} \cup \{5, 7, 9, \ldots\} \cup \{9, 12, 15, 18, 21, 24, 27\}$ *and* $n \geq 3$, $n \notin \{6, 13, 14\}$. *Then there exists a* URD$(\{3, 4\}; 12i)$ *with* $r_3 = 10$ *where* $i = 15n + b + 3$.

Proof The lemma is a special case of Lemma 3.4, let $s = 36$, $j = 10$ and $u = 0$. A URD$(\{3, 4\}; 12b + 36)$ with $r_3 = 10$ exists for $b \in \{0, 1, 4\} \cup \{5, 7, 9, \ldots\} \cup \{9, 12, 15, 18, 21, 24, 27\}$ by the Lemmas 6.1–6.4. Let $m = 3n$ then an IURD$(\{3, 4\}; 12m + 36)$ with $r_3^\circ = 10$ and a hole of size 36 exists by Lemma 6.4 for $n + 1 \geq 4, n + 1 \notin \{14, 15\}$ i.e. $n \geq 3, n \notin \{13, 14\}$. A TD$(6, 3n)$ exists for $3n \geq 5,\ 3n \neq 6,\ 18$ by Theorem 1.3. Both conditions together give: $n \geq 3, n \notin \{6, 13, 14\}$. □

Lemma 6.8 *Let* $0 \leq b \leq 3n$, $b \in \{0, 1, 4\} \cup \{5, 7, 9, \ldots\} \cup \{9, 12, 15, 18, 21, 24, 27\}$, $3 \leq c \leq n$, $c \notin \{13, 14\}$ *and* $n \geq 3$, $n \notin \{5, 6, 10, 13, 14, 20\}$. *Then there exists a* URD$(\{3, 4\}; 12i)$ *with* $r_3 = 10$ *where* $i = 15n + b + 3c + 3$.

Proof The lemma is a special case of Lemma 3.4, let $s = 36$, $j = 10$ and $u = 1$. The condition on b is as in Lemma 6.7. Let $m = 3n$ then an IURD$(\{3, 4\}; 12m + 36)$ with $r_3^\circ = 10$ and a hole of size 36 exists by Lemma 6.4 for $n + 1 \geq 4, n + 1 \notin \{14, 15\}$ i.e. $n \geq 3, n \notin \{13, 14\}$. A TD$(7, 3n)$ exists for $3n \geq 7, 3n \notin \{15, 18, 30, 60\}$ by Theorem 1.3. Both conditions together give $n \geq 3, n \notin \{5, 6, 10, 13, 14, 20\}$. Let $a = 3c$ then it follows how above that an IURD$(\{3, 4\}; 12a + 36)$ with $r_3^\circ = 10$ and a hole of size 36 exists for $c \geq 3, c \notin \{13, 14\}$. □

With $b \in \{0, 1, 5\}$ and $c \in \{3, 4, 5, 6, 7\}$ all residue classes modulo 15 are covered.

Lemma 6.9 *Let* $b \in \{0, 1, 5\}$, $c \in \{3, 4, 5, 6, 7\}$ *and* $n \geq 3, n \notin \{5, 6\}$. *There exists a* URD$(\{3, 4\}; 12i)$ *with* $r_3 = 10$, *where* $i = 15n + b + 3c + 3$, *which means* $\{i : i > 15 \cdot 6 + 5 + 3 \cdot 7 + 3 = 119\} \subset N$.

Proof For $n \notin \{10, 13, 14, 20\}$ all the conditions of Lemma 6.8 are fulfilled. For $n \in \{10, 13, 20\}$ we have $i = 15n + b + 3c + 3 = 15(n - 1) + (b + 12) + 3(c + 1) + 3, b \in \{0, 1, 5\}, c \in \{3, 4, 5, 6, 7\}$ and for the right side of the above equation all the conditions of Lemma 6.8 are fulfilled. For $n = 14$ we have $i = 15n + b + 3c + 3 = 15(n - 2) + (b + 24) + 3(c + 2) + 3, b \in \{0, 1, 5\}, c \in \{3, 4, 5, 6, 7\}$ and for the right side of the above equation all conditions are again fulfilled in Lemma 6.8. □

Lemma 6.10 *We have* $\{43, 47, 67, 83, 115\} \subset N$.

Proof Take in Lemma 6.5, where $i = 10n + b + 2$, the following values for n and b:

n	b	i
4	1	43
4	5	47
6	5	67
8	1	83
10	13	115

□

Lemma 6.11 *We have* $\{53, 69, 71, 73, 77, 89, 95, 97, 101, 103, 107, 109\} \subset N$.

Proof Take in Lemma 6.6, where $i = 10 \cdot n + b + 2c + 2$, the following values for n, b and c:

n	b	c	i
4	5	3	53
6	1	3	69
6	1	4	71
6	1	5	73
6	5	5	77
8	1	3	89
8	1	6	95
8	1	7	97
8	5	7	101
8	13	4	103
8	13	6	107
8	13	7	109

□

Lemma 6.12 *We have* $\{55, 85, 113, 119\} \subset N$.

Proof Take in Lemma 6.7, where $i = 15 \cdot n + b + 3$, the following values for n and b:

n	b	i
3	7	55
5	7	85
7	5	113
7	11	119

□

Lemma 6.13 *We have* $\{61, 79\} \subset N$.

Proof Take in Lemma 6.8, where $i = 15 \cdot n + b + 3c + 3$, the following values for n, b and c:

n	b	c	i
3	4	3	61
4	7	3	79

□

Lemma 6.14 *There exists a* URD$(3, 4, 12 \cdot 37)$ *and a* URD$(\{3,\ 4\};\ \ 12 \cdot 65)$ *both with* $r_3 = 10$, *i.e.* $\{37,\ 65\} \subset N$.

Proof There exist 5-GDDs of types $24^5 20^1$ and $40^5 52^1$ by Theorem 1.2. Apply Construction 3.3 with weight 3 and 4-frames of type 3^5, given in Theorem 1.6. The results are 4-frames of types $72^5 60^1$ and $120^5 156^1$. There exists an IURD$(\{3,\ 4\};\ 72 + 24)$ and an IURD$(\{3,\ 4\};\ 120 + 24)$ both with $r_3^\circ = 10$ and a hole of size 24 by Lemma 6.3. Adjoin 24 infinite points to the frames and fill all groups of size 72 and 120 with the appropriate IURD, with the infinite points forming the hole. Fill the group of size 60 together with the infinite points with a URD$(\{3,\ 4\};\ 84)$ with $r_3 = 10$, which is given in Lemma 6.2. Fill the group of size 156 together with the infinite points with a URD$(\{3,\ 4\};\ 180)$ with $r_3 = 10$, which is given in Lemma 6.4. □

Lemma 6.15 *There exists a* URD$(\{3,\ 4\};\ 12 \cdot 59)$ *with* $r_3 = 10$*, i.e.* $59 \in N$.

Proof There exists a 5-GDD of type $36^5 44^1$ by Theorem 1.2. Apply Construction 3.3 with weight 3 and 4-frames of type 3^5, given in Theorem 1.6. The result is a 4-frame of type $108^5 \cdot 132^\circ$. There exists an IURD$(\{3,\ 4\};\ 108 + 36)$ with $r_3^\circ = 10$ and a hole of size 36 by Lemma 6.4. Adjoin 36 infinite points to the frame and fill all groups of size 108 with this IURD, with the infinite points forming the hole. Fill the group of size 132 together with the infinite points with a URD$(\{3,\ 4\};\ 168)$ with $r_3 = 10$, which exists by Lemma 6.3. □

Lemma 6.16 *There exists a* URD$(\{3,\ 4\};\ 12 \cdot 45)$ *with* $r_3 = 10$.

Proof There exists a TD(7, 7) by Theorem 1.3. Truncate a group of this design to size 1. Use one of the truncated points to redefine the groups. This gives a $\{6,\ 7\}$ -GDD of type $6^7 1^1$. Apply Construction 3.3 with weight 12 and 4-frames of types 12^6 and 12^7, which exist by Theorem 1.6. The result is a 4-frame of type $72^7 12^1$. There exists an IURD$(\{3, 4\};\ 96)$ with $r_3^\circ = 10$ and a hole of size 24 by Lemma 6.3. Adjoin 24 infinite points to the frame and fill all groups of size 72 with this IURD, with the infinite points forming the hole. Fill the group of size 12 together with the infinite points with a URD$(\{3,4\};\ 36)$ with $r_3 = 10$. This gives a URD$(\{3,\ 4\};\ 12 \cdot 45)$ with $r_3 = 10$. □

All lemmas of this section result in:

Theorem 6.17 *There exists a* $URD(\{3,\ 4\};\ 12\,n)$ *with* $r_3 = 10$ *if, and only if,* $n \geq 2$*, and possibly excepting the following 12 values:* $n \in \{5, 6, 9, 11, 13, 17, 19, 23, 25, 29, 31, 41\}$.

7 Results for URDs with exactly 7 parallel classes with blocks of size 4

Lemma 7.1 *There exists a* URD$(\{3,\ 4\};\ 24\,i)$*, with* $r_4 = 7$ *for all integers* $i > 0$ *and an* IURD$(\{3, 4\};\ 24\,i)$ *with* $r_4^\circ = 7$ *and a hole of size* 24 *for all* $i \geq 3$.

Proof A URD$(\{3,\ 4\};\ 24)$ with $r_3 = 1,\ r_4 = 7$ exists by Theorem 1.11. Apply this design in Lemma 3.6. A URD$(\{3,\ 4\};\ 48)$ with $r_4 = 7$ exists by Lemma 2.4. □

Lemma 7.2 *T here exists a* URD$(\{3,\ 4\};\ 36\,i)$*, with* $r_4 = 7$ *for all integers* $i > 0$ *and an* IURD$(\{3, 4\};\ 36\,i)$ *with* $r_4^\circ = 7$ *and a hole of size* 36 *for all* $i \geq 3$.

Proof A URD$(\{3,\ 4\};\ 36)$ with $r_4 = 7$ exists by Lemma 2.4. Apply this design in Lemma 3.6. A URD$(\{3,\ 4\};\ 72)$ with $r_4 = 7$ exists by Lemma 7.1. □

Lemma 7.3 *There exists a* URD$(\{3,\ 4\};\ 84\,i)$ *with* $r_4 = 7$ *or* 9 *for all integers* $i > 0$.

Proof Take as master designs URD$(\{3,\ 4\};\ 12\,i)$ with $r_4 = 1, i = 1, 2, \ldots$, which exist by Theorem 1.10, and expand all points with 7. It is well-known that an RPBD$(3;\ 21)$, an RPBD$(4;\ 28)$, an RTD$(3,\ 7)$ and an RTD$(4,\ 7)$ exist.

We first deal with the case $r_4 = 7$. For only one parallel class with $k = 3$ fill each expanded block with an RPBD$(3;\ 21)$, which also fills allpairs in the expanded groups. For all other parallel classes with $k = 3$ fill each expanded block with an RTD$(3,\ 7)$. For the only parallel class with $k = 4$ fill each expanded block with an RTD$(4,\ 7)$, which gives seven parallel classes with $k = 4$.

Now the case $r_4 = 9$. For all parallel classes with $k = 3$ fill each expanded block with an RTD$(3,\ 7)$. For the only parallel class with $k = 4$ fill each expanded block with an RPBD$(4;\ 28)$, which also fills allpairs in the expanded groups and results in nine parallel classes with $k = 4$. □

Lemma 7.4 *There exists a* URD$(\{3,\ 4\};\ v)$ *with* $r_4 = 7$ *for all* $v \equiv 12 \pmod{72}$ *except* $v = 12$ *and except possibly when* $v = 228,\ 372,\ 444$.

Proof A URD$(\{3,\ 4\};\ 132)$ with $r_4 = 7$ exists by Lemma 2.4, $132 = v_0 = 9r_4+6j+9$ with $j = 10$. By Lemma 3.8 there exists a $\{3,\ 4\}$ -frame of type $(24; 3^{12})^t(v_0 - 9; 3^{3(r_4+j)}, 4^{r_4})^1$ for all $t \equiv 1 \pmod 3$ with $t \geq 1 + \frac{3(r_4+j)}{4} = 13.75$. Let $t = 13 + 3i, i \geq 1$. Therefore by Lemma 3.7 there exists a URD$(\{3,\ 4\};\ 24(13 + 3i) + 132)$ with $r_4 = 7$, that means there exist all URD$(\{3,\ 4\};\ 444 + 72i)$ with $r_4 = 7, i \geq 1$. Due to the third condition of Theorem 1.9 a URD$(\{3,\ 4\};\ 12)$ with $r_4 = 7$ cannot exist. A URD$(\{3,\ 4\};\ 84)$ with $r_4 = 7$ exists by Lemma 7.3. A URD$(\{3,\ 4\};\ 156)$ with $r_4 = 7$ exists by Lemma 2.4. An RGDD$(3,\ 60;\ 300)$ exists by Theorem 1.4. Filling the groups with a URD$(\{3,\ 4\};\ 60)$ with $r_4 = 7$, which exists by Lemma 2.4, results in a URD$(\{3,\ 4\};\ 300)$ with $r_4 = 7$. □

Lemma 7.5 *There exist a* URD$(\{3,\ 4\};\ v)$ *with* $r_4 = 7$ *for all* $v \equiv 60 \pmod{72}$ *except possibly when* $v = 276,\ 348$.

Proof A URD$(\{3,\ 4\};\ 108)$ with $r_4 = 7$ exists by Lemma 7.2, $108 = v_0 = 9r_4+6j+9$ with $j = 6$. By Lemma 3.8 there exists a $\{3,\ 4\}$ -frame of type $(24; 3^{12})^t(v_0 - 9; 3^{3(r_4+j)}, 4^{r_4})^1$ for all $t \equiv 1 \pmod 3$ with $t \geq 1 + \frac{3\cdot(r_4+j)}{4} = 10.75$. Let $t = 10 + 3i, i \geq 1$. Therefore by Lemma 3.7 there exists a URD$(\{3,\ 4\};\ 24(10 + 3i) + 108)$ with $r_4 = 7, i \geq 1$, which means that a URD$(\{3,\ 4\};\ 348 + 72i)$ with $r_4 = 7, i \geq 1$ exist. A URD$(\{3,\ 4\};\ 60)$ with $r_4 = 7$, a URD$(\{3,\ 4\};\ 132)$ with $r_4 = 7$ and a URD$(\{3,\ 4\};\ 204)$ with $r_4 = 7$ exist by Lemma 2.4. □

Lemma 7.6 *Let* $0 \leq b \leq m$, $b \in \{0, 1, \ldots, 12\}$ *and* $m \in \{4, 8, 12\}$. *Then there exists a* URD$(\{3,\ 4\};\ 12 \cdot i)$ *with* $r_4 = 7$ *where* $i = 4m + b + 2$.

Proof This Lemma is a special case of Lemma 3.5, let $s = 24$, $j = 7$ and $u = 0$. A URD$(\{3,\ 4\};\ 12\,b + 24)$ with $r_4 = 7$ exists for $b \in \{0, 1, \ldots, 10\}$ because in the above lemmas the smallest exception is $156 = 13 \cdot 12$. An IURD$(\{3,\ 4\};\ 12\,m + 24)$ with $r_4^\circ = 7$ and a hole of size 24 exists by Lemma 7.1 for $m \in \{4, 6, 8, \ldots\}$. A TD$(5,\ m)$ exists for $m \geq 4,\ m \notin \{6, 10\}$. Both conditions together give: $m \in \{4, 8, 12\}$. □

Lemma 7.7 *There exist* URD$(\{3,\ 4\};\ 228)$ *and* URD$(\{3,\ 4\};\ 444)$ *both with* $r_4 = 7$.

Proof Take in Lemma 7.6 the following values for m and b :

m	b	i
4	1	19
8	3	37

□

Lemma 7.8 *There exists a* URD$(\{3,\ 4\};\ v)$ *with* $r_4 = 7$ *for* $v \in \{276,\ 348,\ 372\}$.

Proof There exist 4-GDDs of types $12^4 15^1$, $18^4 9^1$ and $18^4 15^1$ by Theorem 1.1. Apply Construction 3.3 with weight 4 and 3-frames of type 4^4, which are given in Theorem 1.5. The result are 3-frames of types $48^4 60^1$, $72^4 36^1$ and $72^4 60^1$. There exists an IURD$(\{3,\ 4\};\ 48 + 24)$ and an IURD$(\{3,\ 4\};\ 72+24)$ both with $r_4^\circ = 7$ and a hole of size 24 by Lemma 7.1. Adjoin 24 infinite points to the above frames and fill all groups of size 48 or 72 with the appropriate IURD, with the infinite points forming the hole. Fill the group of size 60 together with the infinite points with a URD$(\{3,\ 4\};\ 84)$ with $r_4 = 7$, which is given in Lemma 7.3. Fill the group of size 36 together with the infinite points with a URD$(\{3,\ 4\};\ 60)$ with $r_4 = 7$, which is given in Lemma 2.4. □

All lemmas of this section give:

Theorem 7.9 *There exists a* URD$(\{3,\ 4\};\ v)$ *with* $r_4 = 7$ *if, and only if,* $v \equiv 0 \pmod{12}$ *except* $v = 12$.

8 Results for URDs with exactly 9 parallel classes with blocks of size 4

Lemma 8.1 *There exists a* URD$(\{3,\ 4\};\ 36 \cdot i)$ *with* $r_4 = 9$ *for all integers* $i \geq 1$ *and an* IURD$(\{3,\ 4\};\ 36i)$ *with* $r_4^\circ = 9$ *and a hole of size* 36 *for* $i \geq 3$.

Proof A URD$(\{3,\ 4\};\ 36)$ with $r_4 = 9$ exists by Lemma 2.4. A URD$(\{3,\ 4\};\ 72)$ with $r_4 = 9$ exists by Lemma 2.4. For $i \geq 3$, integer, take as master designs 3-RGDDs of type 36^i, RGDD$(3, 36; 36i)$, which exist by Theorem 1.4. Filling the groups with a URD$(\{3,\ 4\};\ 36)$ with $r_4 = 9$ results in a URD$(\{3,\ 4\};\ 36\ i)$ with $r_4 = 9$. By not filling one group we obtain the IURDs. □

Lemma 8.2 *There exists an* LURD$_4(\{3,\ 4\};\ 24i)$ *with* $r_4 = 8$ *and also a* URD$(\{3,\ 4\};\ 96i)$ *with* $r_4 = 9$ *for all integers* $i \geq 1$.

Proof There exists a URD$(\{3,\ 4\};\ 12i)$ with $r_4 = 1$ for $i \geq 1$ by Theorem 1.10. Take this design as master design and expand all points with two. For only one parallel class with $k = 3$ fill each expanded block with an LRPBD$_4(3;\ 6)$, which also fills allpairs in the expanded groups. For all other parallel classes with $k = 3$ fill each expanded block with an LRTD$_4(3,\ 2)$. For the only parallel class with $k = 4$ fill each expanded block with an LRTD$_4(4,\ 2)$, which gives 8 parallel classes with $k = 4$. All above labeled designs are given in the Online Resource. Hence, an LURD$_4(\{3,\ 4\};\ 24\ i)$ with $r_4 = 8$ is constructed. By Corollary 2.2 there exists a URD$(\{3,\ 4\};\ 96\ i)$ with $r_4 = 9$ for all $i \geq 1$. □

Lemma 8.3 *There exists a* URD$(\{3,\ 4\};\ v)$ *with* $r_4 = 9$ *for all* $v \equiv 12 \pmod{72}$ *except* $v = 12$ *and except possibly when* $v = 372$.

Proof A URD$(\{3,\ 4\};\ 132)$ with $r_4 = 9$ exists by Lemma 2.4, $132 = v_0 = 9r_4 + 6j + 9$ with $j = 7$. By Lemma 3.8 there exists a $\{3,\ 4\}$ -frame of type $(24;\ 3^{12})^t(v_0 - 9;\ 3^{3(r_4+j)},\ 4^{r_4})^1$ for all $t \equiv 1 \pmod{3}$ with $t \geq 1 + \frac{3(r_4+j)}{4} = 13$. Let $t = 10 + 3i$ with $i = 1, 2, \ldots$. Therefore by Lemma 3.7 there exists a URD$(\{3,\ 4\};\ 24(10 + 3i) + 132)$ with $r_4 = 9$, which means that a URD$(\{3,\ 4\};\ 372 + 72i)$ with $r_4 = 9$ exists for all $i \geq 1$.

Due to the third condition of Theorem 1.7 a URD$(\{3,\ 4\};\ 12)$ with $r_4 = 9$ cannot exist. A URD$(\{3,\ 4\};\ 84)$ with $r_4 = 9$ exists by Lemma 7.3. A URD$(\{3,\ 4\};\ 156)$ and a URD$(\{3,\ 4\};\ 228)$ both with $r_4 = 9$ exist by Lemma 2.4. An RGDD$(3,\ 60;\ 300)$ exists by Theorem 1.4. Filling the groups with a URD$(\{3,\ 4\};\ 60)$ with $r_4 = 9$, which exists by Lemma 2.4, results in a URD$(\{3,\ 4\};\ 300)$ with $r_4 = 9$. □

Lemma 8.4 *There exists a* URD$(\{3,\ 4\};\ v)$ *with* $r_4 = 9$ *for all* $v \equiv 24 \pmod{72}$ *except* $v = 24$ *and except possibly when* $v = 312,\ 456$.

Proof A URD$(\{3,\ 4\};\ 48)$ with $r_4 = 9$ exists by Lemma 2.4 and therefore there exists also a URD$(\{3,\ 4\};\ 144)$ with $r_4 = 9$ by Lemma 3.6, $144 = v_0 = 9r_4 + 6j + 9$ with $j = 9$. By Lemma 3.8 there exists a $\{3,\ 4\}$ -frame of type $(24;\ 3^{12})^t(v_0 - 9;\ 3^{3(r_4+j)},\ 4^{r_4})^1$ for all $t \equiv 1 \pmod{3}$ with $t \geq 1 + \frac{3(r_4+j)}{4} = 14.5$. Let $t = 13 + 3i$ with $i \geq 1$. Therefore by Lemma 3.7 there exist URD$(\{3,\ 4\};\ 24 \cdot (13 + 3i) + 144)$ with $r_4 = 9$, which means that a URD$(\{3,\ 4\};\ 456 + 72i)$ with $r_4 = 9$ exists for all $i \geq 1$.

For $v = 24$ and $r_4 = 9$ would be r_3 negative and therefore there is no URD($\{3, 4\}$; 24) with $r_4 = 9$. A URD($\{3, 4\}$; 96) with $r_4 = 9$ exists by Lemma 8.2. A URD($\{3, 4\}$; 168) with $r_4 = 9$ exists by Lemma 7.3. An RGDD(3, 60; 240) exists by Theorem 1.4. Filling the groups with URD($\{3, 4\}$; 60) with $r_4 = 9$, which exists by Lemma 2.4, results in a URD ($\{3, 4\}$; 240) with $r_4 = 9$. A URD($\{3, 4\}$; 384) with $r_4 = 9$ exists by Lemma 8.2. □

Lemma 8.5 *There exists a* URD($\{3, 4\}$; v) *with* $r_4 = 9$ *for all* $v \equiv 48 \pmod{72}$ *except possibly when* $v = 264$.

Proof A URD($\{3, 4\}$; 96) with $r_4 = 9$ exists by Lemma 8.2, taking $96 = v_0 = 9r_4 + 6j + 9$ with $j = 1$. By Lemma 3.8 there exists a $\{3, 4\}$ -frame of type $(24; 3^{12})^t(v_0 - 9; 3^{3(r_4+j)}, 4^{r_4})^1$ for all $t \equiv 1 \pmod 3$ with $t \geq 1 + \frac{3(r_4+j)}{4} = 8.5$. Let $t = 7 + 3i$ with $i \geq 1$. Therefore by Lemma 3.7 there exists a URD($\{3, 4\}$; $24(7 + 3i) + 96$) with $r_4 = 9$, which means that a URD($\{3, 4\}$; $264 + 72i$) with $r_4 = 9$ exists for all $i \geq 1$.

A URD($\{3, 4\}$; 48) with $r_4 = 9$ exists by Lemma 2.4. A $\{3, 4\}$-LRGDD$_5$ of type 3^8 with $r_3 = 39$ and $r_4 = 9$ is given in the Online Resource. Therefore, there exists a $\{3, 4\}$-RGDD of type 15^8 with $r_3 = 39$ and $r_4 = 9$. Filling all groups with a RPBD(3; 15) results in a URD ($\{3, 4\}$; 120) with $r_4 = 9$. A URD($\{3, 4\}$; 192) with $r_4 = 9$ exists by Lemma 8.2. □

Lemma 8.6 *There exists a* URD($\{3, 4\}$; v) *with* $r_4 = 9$ *for all* $v \equiv 60 \pmod{72}$ *except possibly when* $v = 276$.

Proof A URD($\{3, 4\}$; 108) with $r_4 = 9$ exists by Lemma 8.1, taking $108 = v_0 = 9r_4 + 6j + 9$ with $j = 3$. By Lemma 3.8 there exists a $\{3, 4\}$ -frame of type $(24; 3^{12})^t(v_0 - 9; 3^{3(r_4+j)}, 4^{r_4})^1$ for all $t \equiv 1 \pmod 3$ with $t \geq 1 + \frac{3(r_4+j)}{4} = 10$. Let $t = 7 + 3i$with i = 1,2,…. Therefore by Lemma 3.7 there exists a URD($\{3, 4\}$; $24(7+3i)+108$) with $r_4 = 9$, which means that a URD($\{3, 4\}$; $276 + 72i$) with $r_4 = 9$, $i = 1, 2, \ldots$ exists.

There exist a URD($\{3, 4\}$; 60), URD($\{3, 4\}$; 132) and a URD($\{3, 4\}$; 204) all with $r_4 = 9$ by Lemma 2.4. □

Lemma 8.7 *There exists a* URD($\{3, 4\}$; 372) *with* $r_4 = 9$.

Proof There exists a 4-GDD of type $18^4 12^1$ by Theorem 1.1. Apply Construction 3.3 with weight 4 and 3-frames of type 4^4, which are given in Theorem 1.5. The result is a 3-frame of type $72^4 48^1$. There exist an IURD($\{3, 4\}$; $72 + 36$) with $r_4^\circ = 9$ and a hole of size 36. Adjoin 36 infinite points to the frame and fill all groups of size 72 with this IURD, with the infinite points forming the hole. Fill the group of size 48 together with the infinite points with a URD($\{3, 4\}$; 84) with $r_4 = 9$, which exists by Lemma 7.3. □

Lemma 8.8 *There exists a* URD($\{3, 4\}$; 456) *with* $r_4 = 9$.

Proof There exists a 4-GDD of type $24^4 6^1$ by Theorem 1.1. Apply Construction 3.3 with weight 4 and 3-frames of type 4^4, which are given in Theorem 1.5. The result is a 3-frame of type $96^4 24^1$. There exist a 3-RGDD of type 48^3 by Theorem 1.4 and a URD($\{3, 4\}$; 48) with $r_4 = 9$ by Lemma 2.4. Filling 2 groups of the RGDD with this URD results in an IURD($\{3, 4\}$; $96 + 48$) with $r_4^\circ = 9$ and a hole of size 48. Adjoin 48 infinite points to the frame and fill all groups of size 96 with this IURD, with the infinite points forming the hole. Fill the group of size 24 together with the infinite points with a URD($\{3, 4\}$; 72) with $r_4 = 9$, which exists by Lemma 2.4. □

Lemma 8.9 *There exists a* URD({3, 4}; 264) *and a* URD({3, 4}; 312) *both with* $r_4 = 9$.

Proof There exist uniform 4-RGDDs of types 2^{22} and 4^{13} by Theorem 1.4, which are used as master designs. By Theorem 1.4 there exists a 3-RGDD of type 6^4, which is used as first ingredient design. In the Online Resource is given a uniform $\{3, 4\} - \text{LRGDD}_2$ of type 3^4 with $r_3 = 3$ and $r_4 = 4$. By Theorem 2.1 we obtain a uniform $\{3, 4\} - \text{RGDD}$ of type 6^4 with $r_3 = 3$ and $r_4 = 4$, which is used as second ingredient design. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Two parallel classes are expanded with the second ingredient design. All other parallel classes have to expand with the first ingredient design, resulting in a uniform $\{3, 4\} - \text{RGDD}$ of type 12^{22} and a uniform $\{3, 4\} - \text{RGDD}$ of type 24^{13} both with $r_4 = 8$. We fill all groups of size 12 with a URD({3, 4}; 12) with $r_4 = 1$ and all groups of size 24 with a URD({3, 4}; 24) with $r_4 = 1$, which results in the desired designs. □

All lemmas of this section result in:

Theorem 8.10 *There exists a* URD({3, 4}; v) *with* $r_4 = 9$ *if, and only if,* $v \equiv 0 \pmod{12}$ *except* $v = 12, 24$ *and except possibly when* $v = 276$.

9 All admissible URD({3, 4}; v) for many values v

Lemma 9.1 *There exist uniform*
3-RGDD *of type* 12^4 *with* $r_3 = 18$ *(and* $r_4 = 0$*),*
$\{3, 4\}$*-RGDD of type* 12^4 *with* $r_3 = 15$ *and* $r_4 = 2$*,*
$\{3, 4\}$*-RGDD of type* 12^4 *with* $r_3 = 12$ *and* $r_4 = 4$*,*
$\{3, 4\}$*-RGDD of type* 12^4 *with* $r_3 = 9$ *and* $r_4 = 6$*,*
$\{3, 4\}$*-RGDD of type* 12^4 *with* $r_3 = 6$ *and* $r_4 = 8$ *and*
4*-RGDD of type* 12^4 *with (*$r_3 = 0$ *and)* $r_4 = 12$.

Proof The 3-RGDD and the 4-RGDD exist by Theorem 1.4. Uniform $\{3, 4\} - \text{LRGDD}_4$ of type 3^4 with $(r_3, r_4) \in \{(15, 2), (12, 4), (9, 6), (6, 8)\}$ are all given in the Online Resource, therefore the assertion follows with Theorem 2.1. □

Theorem 9.2 *If a* 4-RGDD *of type* h^u *exists, then there exists a* URD({3, 4}; $12hu$) *with* $r_3 = 1, 4, \ldots, 6hu - 2$, *which means that for* $v = 12hu$ *all admissible* $\{3, 4\} - \text{URDs}$ *exist.*

Proof URDs with $r_3 = 1$ exist by Theorem 1.11. We take the 4-RGDD of type h^u as master design and all designs of Lemma 9.1 as ingredient designs. We expand all points of the master design 12 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 6, 9, 12, 15 or 18 3-pcs. We obtain a $\{3, 4\}$-RGDD of type $(12h)^u$ with $r_3 = 0, 6, 9, \ldots, 6h(u-1)$, as we fill all parallel classes appropriately. Now we fill all groups. At first, we fill all groups with a URD({3, 4}; $12h$) with $r_4 = 1$ and $r_3 = 6h - 2$, which exists by Theorem 1.10, and obtain a URD({3, 4}; $12hu$) with $r_3 = 6h - 2 + 6, \ldots, 6hu - 2$. Secondly, we can fill all groups with a URD({3, 4}; $12h$) with $r_3 = 4$, which exists by Theorem 4.1, and obtain a URD({3, 4}; $12hu$) with $r_3 = 4, 10, 13, \ldots, 6\,h(u-1) + 4$. A URD({3, 4}; $12hu$) with $r_3 = 7$ exists by Theorem 5.16, because the product hu from a 4-RGDD of type h^u cannot be an exceptional value in Theorem 5.16. □

Theorem 9.3 *For* $v \equiv 0 \pmod{48}$, *all admissible* URD$(\{3,\ 4\};\ v)$ *exist.*

Proof There exist 4-RGDDs of type g^4 by Theorem 1.4 for all integers $g \geq 1$, except when $g \in \{2, 3, 6\}$ and except possibly when $g = 10$. Therefore, for $v \equiv 0 \pmod{48}$, all admissible URD$(\{3,\ 4\};\ v)$ exist by Theorem 9.2, except possibly when $v = 96,\ 144, 288$ or 480.

There exists a 4-RGDD of type 4^{10} by Theorem 1.4, and therefore, all admissible URD$(\{3,\ 4\};\ 480)$ exist by Theorem 9.2.

All RGDDs in the remain of this proof have only uniform pcs. We take a 4-RGDD of type 3^8 as master design and all designs of Lemma 9.1 as ingredient designs. We assign weight 12 to all points of the master design. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 2, 4, 6, 8 or 12 4-pcs. We obtain a $\{3,\ 4\}$-RGDD of type 36^8 with $r_4 = 0, 2, 4, \ldots, 80, 84$ if we fill all parallel classes appropriately. Only $r_4 = 82$ is not combinable. There exist all admissible URD$(\{3,\ 4\};\ 36)$ by Theorems 1.4, 1.13, 4.1, 5.16, 6.17, 7.9, 8.10. Now, we fill all groups appropriately with the same such design. The result is a URD$(\{3,\ 4\};\ 288)$ with $r_4 = 1 \pmod 2$, $1 \leq r_4 \leq 95$. Each such r_4 is combinable.

We take a $\{3, 4\}$-RGDD of type 3^4 as master design and all designs of Lemma 9.1 as ingredient designs. All points of the master design are given weight 12. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. The only 4-pc of the master design results in 0, 2, 4, 6, 8 or 12 4-pcs. We obtain a $\{3, 4\}$-RGDD of type 36^4 with $r_4 = 0, 2, 4, 6, 8, 12$. Only $r_4 = 10$ is not combinable. There exist all admissible URD$(\{3,\ 4\};\ 36)$ by Theorems 1.4, 1.13, 4.1, 5.16, 6.17, 7.9, 8.10. Now, we fill all groups appropriately with the same such design. The result is a URD$(\{3,\ 4\};\ 144)$ with $r_4 = 1 \pmod 2$, $1 \leq r_4 \leq 23$. Each such r_4 is combinable. A uniform $\{3, 4\} - LRGDD_{12}$ of type 3^4 with $r_4 = 24$ is given in the Online Resource. Therefore, there exists a uniform $\{3, 4\}$-RGDD of type 36^4 with $r_4 = 24$ by Theorem 2.1. By filling all groups with the same URD$(\{3,\ 4\};\ 36)$, we obtain a URD$(\{3,\ 4\};\ 144)$ with $r_4 = 1 \pmod 2$, $25 \leq r_4 \leq 35$. There exists a 4-RGDD of type 36^4 by Theorem 1.4 which has 36 4-pcs. By filling all groups with the same URD$(\{3,\ 4\};\ 36)$, we obtain a URD$(\{3,\ 4\};\ 144)$ with $r_4 = 1 \pmod 2$, $37 \leq r_4 \leq 47$.

There exist a URD$(\{3,\ 4\};\ 96)$ with $r_4 = 1, 3, 5, 7, 9$ by Theorems 1.4, 1.13, 7.9, 8.10. We take a $\{3, 4\}$-RGDD of type 3^4 as master design and a 3-RGDD of type 8^3 as well as a 4-RGDD of type 8^4 as ingredient designs (Theorem 1.4). All points of the master design are assigned weight 8. The only 4-pc of the master design results in 8 4-pcs. We obtain a $\{3,4\}$-RGDD of type 24^4 with $r_4 = 8$. There exist all admissible URD$(\{3, 4\}; 24)$ by Theorems 1.4, 1.13, 7.9. Now, we fill all groups appropriately with the same such design. The result is a URD$(\{3, 4\}; 96)$ with $r_4 = 1 \pmod 2$, $9 \leq r_4 \leq 15$. A uniform $\{3, 4\}$- LRGDD_8 of type 3^4 with $r_4 = 16$ is given in the Online Resource. Therefore, there exists a uniform $\{3,4\}$-RGDD of type 24^4 with $r_4 = 16$ by Theorem 2.1. By filling all groups with an appropriate URD$(\{3, 4\}; 24)$, we obtain a URD$(\{3, 4\}; 96)$ with $r_4 = 1 \pmod 2$, $17 \leq r_4 \leq 23$. There exists a URD$(\{3, 4\}; 96)$ with $r_4 = 1 \pmod 2$, $25 \leq r_4 \leq 31$ by Theorems 1.4, 4.1, 5.16, 6.17. □

10 Three new 4-RGDDs

In this section, we improve the result in Theorem 1.4 by eliminating three possible exceptions.

Lemma 10.1 *There exists a 4-RGDD of type* 2^{142}.

Proof There exists a 5-GDD of type $16^5 12^1$ by Theorem 1.2. Apply Construction 3.3 with weight 3 and a 4-frame of type 3^5, which exists by Theorem 1.6. The result is a 4-frame of type $48^5 36^1$. Adjoin 8 infinite points to this frame and fill all groups of size 48 with an IURD({2,4}; 48+8) with $r_2^\circ = 1$ (see [17, Lemma 2.5]), with the infinite points forming the hole. Fill the group of size 36 together with the infinite points with a URD({2,4}; 44) with $r_2 = 1$, which exists by Theorem 1.4. The result is a URD({2,4}; 284) with $r_2 = 1$, which gives a 4-RGDD of type 2^{142}. □

Lemma 10.2 *There exists a 4-RGDD of type* 2^{346}.

Proof There exists a TD(7,8) by Theorem 1.3. Truncate a group of this design to size 7. This gives a {6,7}-GDD of type $8^6 7^1$. Apply Construction 3.3 with weight 12 and 4-frames of types and 12^6 and 12^7, which exist by Theorem 1.6. The result is a 4-frame of type $96^6 84^1$.

Fill three of the groups of a 4-RGDD of type 32^4, which exists by Theorem 1.4, with a URD({2,4};32) with $r_2 = 1$, which exists by Theorem 1.4, results in an IURD({2,4}; 96+32) with $r_2^\circ = 1$ and a hole of size 32. Adjoin 32 infinite points to above frame and fill all groups of size 96 with this IURD, with the infinite points forming the hole. Fill the group of size 84 together with the infinite points with a URD({2, 4};116) with $r_2 = 1$, which exists by Theorem 1.4. The result is a URD({2, 4};692) with $r_2 = 1$, which gives a 4-RGDD of type 2^{346}. □

Lemma 10.3 *There exists a 4-RGDD of type* 6^{54}.

Proof Let the point set be $(Z_{106} \cup \{x, y\}) \times Z_3$, and let the group set be $\{\{j, j+53\} \times Z_3 : j = 0, 1, \ldots, 52\} \cup \{\{x, y\} \times Z_3\}$\{{j, j + 53}. Below are the required base blocks.

{(90,1),(93,1),(95,1),(35,1)}	{(74,1),(54,2),(80,2),(75,1)}	(0,1),(62,1),(77,1),(79,2)}
{(29,1),(39,1),(98,2),(105,2)}	{(44,1),(19,2),(94,1),(87,3)}	{(18,1),(28,2),(34,2),(102,2)}
{(41,1),(60,3),(23,3),(40,3)}	{(104,1),(81,3),(69,3),(11,2)}	{(91,1),(1,3),(58,3),(85,3)}
{(10,1),(78,2),(82,2),(89,3)}	{(59,1),(33,2),(83,3),(68,2)}	{(3,1),(20,3),(32,2),(42,1)}
{(5,1),(9,2),(63,2),(72,2)}	{(96,1),(4,3),(13,1),(53,2)}	{(12,1),(46,1),(100,2),(103,3)}
{(2,1),(76,3),(56,2),(15,3)}	{(88,1),(57,1),(38,2),(99,1)}	{(24,1),(65,3),(97,1),(21,2)}
{(36,1),(49,1),(7,2),(25,2)}	{(45,1),(61,1),(92,2),(37,1)}	{(47,1),(84,2),(6,1),(14,2)}
{(67,1),(31,3),(8,1),(86,1)}	{(101,1),(55,3),(30,3),(16,1)}	{(52,1),(64,2),(22,1),(50,2)}
{(70,1),(66,3),(26,3),(71,2)}	{(27,1),(48,3),(73,2),(y,1)}	{(17,1),(43,3),(51,2),(x,1)}

Here, we first develop these blocks (−, mod 3) to get a parallel class. Then, we develop this parallel class (mod 106, −) to obtain the 4-RGDD of type 6^{54} as required. □

Combining the above lemmas and Theorem 1.4, we have:

Theorem 10.4 *The necessary conditions for the existence of an k -RGDD of type* h^n, RGDD $(k, h; hn)$*, namely,* $n \geq k$, $hn \equiv 0 \pmod{k}$ *and* $h(n-1) \equiv 0 \pmod{k-1}$*, are also sufficient for*

$k = 2$;
$k = 3$*, except for* $(h, n) \in \{(2, 3), (2, 6), (6, 3)\}$*; and for*
$k = 4$*, except for* $(h, n) \in \{(2, 4), (2, 10), (3, 4), (6, 4)\}$ *and possibly excepting:*

5. $h \equiv 2, 10 \pmod{12}$:
   $h = 2$ *and* $n \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 178, 184, 202, 214, 238, 250, 334\}$;
   $h = 10$ *and* $n \in \{4, 34, 52, 94\}$;
   $h \in [14, 454] \cup \{478, 502, 514, 526, 614, 626, 686\}$ *and* $n \in \{10, 70, 82\}$.

6. $h \equiv 6 \pmod{12}$: $h = 6$ *and* $n \in \{6, 68\}$; $h = 18$ *and* $n \in \{18, 38, 62\}$.
7. $h \equiv 9 \pmod{12}$: $h = 9$ *and* $n = 44$.
8. $h \equiv 0 \pmod{12}$: $h = 36$ *and* $n \in \{11, 14, 15, 18, 23\}$.

Acknowledgments The authors thank the referees for their careful reading and many valuable comments and suggestions. Research of the second author was supported by the National Outstanding Youth Science Foundation of China under Grant No. 10825103, National Natural Science Foundation of China under Grant No. 10771193, Specialized Research Fund for the Doctoral Program of Higher Education, Program for New Century Excellent Talents in University, and Zhejiang Provincial Natural Science Foundation of China under Grant No. D7080064.

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