New classes of group divisible designs with block size 4 and group type $g^u m^1$

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Abstract
In this paper, we show that there exist all admissible 4-GDDs of type $g^6 m^1$ for $g \equiv 0 \pmod 6$.
For 4-GDDs of type $g^u m^1$, where g is a multiple of 12, the most values of m are determined.
Particularly, all spectra of 4-GDDs of type $g^u m^1$ are attained, where g is a multiple of 24 or 36. Furthermore, we show that all 4-GDDs of type $g^u m^1$ exist for $g = 10, 20, 28, 84$ with some possible exceptions.

Keywords: Group divisible design; Labeled group divisible design; Resolvable group divisible design; Transversal design

1. Introduction

A *group divisible design* (GDD) with *index* λ is a triple (X, G, B), where X is a set of points, G is a partition of X into groups, and B is a collection of subsets of X called blocks such that any pair of distinct points from X occurs either in some group or in exactly λ blocks, but not both. A $K-\text{GDD}_\lambda$ of type $g_1^{u_1} g_2^{u_2} \ldots g_s^{u_s}$ is a GDD in which each block has a size from set K and in which there are u_i groups of size g_i, $i = 1, 2, \ldots, s$. The notation is similar to [3] [6]. If $\lambda = 1$, the index λ is omitted. If $K = \{k\}$ then the $K-\text{GDD}_\lambda$ is simply denoted by $k-\text{GDD}_\lambda$.

Theorem 1.1 ([4]). *Let* g *and* u *be positive integers. Then there exists a 4-GDD of type* g^u *if, and only if, the conditions in Table 1 are satisfied.*

Table 1
Existence of 4-GDDs of type g^u

g	u	Necessary and sufficient conditions
$\equiv 1, 5 \pmod 6$	$\equiv 1, 4 \pmod{12}$	$u \geq 4$
$\equiv 2, 4 \pmod 6$	$\equiv 1 \pmod 3$	$u \geq 4$, $(g, u) \neq (2, 4)$
$\equiv 3 \pmod 6$	$\equiv 0, 1 \pmod 4$	$u \geq 4$
$\equiv 0 \pmod 6$	no constraint	$u \geq 4, (g, u) \neq (6, 4)$

For a 4-GDD of type $g^u m^1$ with $g, m \geq 0$ and $u \geq 4$ are the necessary conditions summarized in Table 2.

Table 2 ([13])
Necessary existence criteria for a 4-GDD of type $g^u m^1$ with $u \geq 4$

g	u	m	$m_{\min}$	$m_{\max}$
$\equiv 0 \pmod 6$	no conditions	$\equiv 0 \pmod 3$	0	$g(u-1)/2$
$\equiv 1 \pmod 6$	$\equiv 0 \pmod{12}$	$\equiv 1 \pmod 3$	1	$(g(u-1)-3)/2$
	$\equiv 3 \pmod{12}$	$\equiv 1 \pmod 6$	1	$g(u-1)/2$
	$\equiv 9 \pmod{12}$	$\equiv 4 \pmod 6$	4	$g(u-1)/2$
$\equiv 2 \pmod 6$	$\equiv 0 \pmod 3$	$\equiv 2 \pmod 3$	2	$g(u-1)/2$
$\equiv 3 \pmod 6$	$\equiv 0 \pmod 4$	$\equiv 0 \pmod 3$	0	$(g(u-1)-3)/2$
	$\equiv 1 \pmod 4$	$\equiv 0 \pmod 6$	0	$g(u-1)/2$
	$\equiv 3 \pmod 4$	$\equiv 3 \pmod 6$	3	$g(u-1)/2$
$\equiv 4 \pmod 6$	$\equiv 0 \pmod 3$	$\equiv 1 \pmod 3$	1	$g(u-1)/2$
$\equiv 5 \pmod 6$	$\equiv 0 \pmod{12}$	$\equiv 2 \pmod 3$	2	$(g(u-1)-3)/2$
	$\equiv 3 \pmod{12}$	$\equiv 5 \pmod 6$	5	$g(u-1)/2$
	$\equiv 9 \pmod{12}$	$\equiv 2 \pmod 6$	2	$g(u-1)/2$

Theorem 1.2 ([14]). *The necessary conditions of Table 2 for a 4-GDD of type $g^u m^1$ are sufficient for the minimum values of m, except that there is no 4-GDD of type $6^4 0^1$, but a 4-GDD of type $6^4 3^1$, and except possibly for the types $11^{12}2^1$, $17^{12}2^1$, $11^{21}2^1$ and $11^{27}5^1$.*
The necessary conditions of Table 2 for a 4-GDD of type $g^u m^1$ are sufficient for the maximum values of m, except that there is no 4-GDD of type $2^6 5^1$.

Theorem 1.3 ([16], [21]). *There exists a 4-GDD of type* g^4m^1 *if, and only if,* $g \equiv m \equiv 0 \pmod 3$ *and* $0 \le m \le 3g/2$ *except for* $(g, m) = (6, 0)$.

For some small values of g, an almost complete solution was found.

Theorem 1.4 ([22], [17], [13]).

1. A 4-GDD of type $1^u m^1$ *exists if, and only if, either* $u \equiv 0 \pmod{12}$ *and* $m \equiv 1 \pmod 3$, $1 \le m \le ((u-1)-3)/2$; *or* $u \equiv 3 \pmod{12}$ *and* $m \equiv 1 \pmod 6$, $1 \le m \le (u-1)/2$; *or* $u \equiv 9 \pmod{12}$ *and* $m \equiv 4 \pmod 6$, $4 \le m \le (u-1)/2$.

2. There exists a 4-GDD of type $2^u m^1$ *for each* $u \ge 6$, $u \equiv 0 \pmod 3$ *and* $m \equiv 2 \pmod 3$ *with* $2 \le m \le u-1$ *except for* $(u, m) = (6, 5)$ *and possibly excepting* $(u, m) \in \{(21, 17), (33,23), (33, 29), (39, 35), (57, 44)\}$.

3. A 4-GDD of type $3^u m^1$ *exists if, and only if, either* $u \equiv 0 \pmod 4$ *and* $m \equiv 0 \pmod 3$, $0 \le m \le (3(u-1)-3)/2$; *or* $u \equiv 1 \pmod 4$ *and* $m \equiv 0 \pmod 6$, $0 \le m \le 3(u-1)/2$; *or* $u \equiv 3 \pmod 4$ *and* $m \equiv 3 \pmod 6$, $0 < m \le 3(u-1)/2$.

4. There exists a 4-GDD of type $4^u m^1$ *for each* $u \ge 6$, $u \equiv 0 \pmod 3$ *and* $m \equiv 1 \pmod 3$ *with* $1 \le m \le 2(u-1)$.

5. A 4-GDD of type $5^u m^1$ *exists if, and only if, either* $u \equiv 0 \pmod{12}$ *and* $m \equiv 2 \pmod 3$, $2 \le m \le (5(u-1)-3)/2$; *or* $u \equiv 3 \pmod{12}$ *and* $m \equiv 5 \pmod 6$, $5 \le m \le 5(u-1)/2$; *or* $u \equiv 9 \pmod{12}$ *and* $m \equiv 2 \pmod 6$, $2 \le m \le 5(u-1)/2$.

6. There exists a 4-GDd of type $6^u m^1$ *for each* $u \ge 4$ *and* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 3(u-1)$ *except for* $(u, m) = (4, 0)$ *and possibly excepting* $(u, m) \in \{(7, 15), (11,21), (11, 24), (11, 27), (13, 27), (13, 33), (17,39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$.

7. There exists a 4-GDD of type $12^u m^1$ *for each* $u \ge 4$ *and* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 6(u-1)$.

8. A 4-GDD of type $15^u m^1$ *exists if, and only if, either* $u \equiv 0 \pmod 4$ *and* $m \equiv 0 \pmod 3$, $0 \le m \le (15(u-1)-3)/2$; *or* $u \equiv 1 \pmod 4$ *and* $m \equiv 0 \pmod 6$, $0 \le m \le 15(u-1)/2$; *or* $u \equiv 3 \pmod 4$ *and* $m \equiv 3 \pmod 6$, $3 \le m \le 15(u-1)/2$.

A *transversal design* $\mathrm{TD}_\lambda(k,g)$, is equivalent to a $k-\mathrm{GDD}_\lambda$ of type g^k. This means that, each block in a $\mathrm{TD}_\lambda(k,g)$ contains a point from each group. If $\lambda=1$, the index λ is omitted.

Theorem 1.5 ([2]). *A* $\mathrm{TD}(k, g)$ exists in the following cases:

1. $k=4$, $g\geq 3$ *and* $g\neq 6$;
2. $k=5$, $g\geq 4$ *and* $g\notin\{6, 10\}$;
3. $k=6$, $g\geq 5$ *and* $g\notin\{6, 10, 14, 18, 22\}$;
4. $k=7$, $g\geq 7$ *and*
   $g\notin\{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$;
5. $k=8$, $g\geq 7$ *and*
   $g\notin\{10, 12, 14, 15, 18, 20, 21, 22, 26, 28, 30, 33, 34, 35, 38, 39, 42, 44, 45, 51, 52, 54, 58, 60, 62, 66, 68, 74\}$.

In a $K-\mathrm{GDD}_\lambda$, a *parallel class* is a set of blocks, which partitions *X*. If *B* can be partitioned into parallel classes, the $K-\mathrm{GDD}_\lambda$ is called *resolvable* and denoted by $K-\mathrm{RGDD}_\lambda$. A parallel class is called *uniform* if it contains blocks of only one size *k* (*k*-pc). If all parallel classes of a $K-\mathrm{RGDD}_\lambda$ are uniform, the design is called *uniformly resolvable*. The following theorem about RGDDs will be applied later.

Theorem 1.6 ([6], [7], [8], [9], [10], [11], [15], [20], [23], [27], [29], [31], [32]). *The necessary conditions for the existence of a k-RGDD of type* g^u, *namely,* $u\geq k$, $gu\equiv 0 \pmod{k}$ *and* $g(u-1)\equiv 0 \pmod{k-1}$, *are also sufficient for* $k=3$, *except for* $(g,u)\in\{(2,3),(2,6),(6,3)\}$; *and for* $k=4$, *except for* $(g,u)\in\{(2,4),(2,10),(3,4),(6,4)\}$ *and possibly except:*

1. $g\equiv 2, 10 \pmod{12}$:
   $g=2$ *and*
   $u\in\{34,46,52,70,82,94,100,118,130,178,184,202,214,238,250,334\}$;
   $g=10$ *and* $u\in\{4, 34, 52, 94\}$;
   $g\in[14,454]\cup\{478, 502, 514, 526, 614, 626, 686\}$ *and*
   $u\in\{10, 70, 82\}$.
2. $g\equiv 6 \pmod{12}$: $g=6$ *and* $u\in\{6,68\}$;
   $g=18$ *and* $u\in\{18, 38, 62\}$.

3. $g \equiv 9 \pmod{12}$: $g = 9$ *and* $u = 44$.
4. $g \equiv 0 \pmod{12}$: $g = 24$ *and* $u = 23$;
   $g = 36$ *and* $u \in \{11, 14, 15, 18, 23\}$.

A *resolvable transversal design* $\mathrm{RTD}_\lambda(k,g)$, is equivalent to a $k-\mathrm{RGDD}_\lambda$ of type g^k. A *double group divisible design* (DGDD) is a quadruple $(X, \mathbf{G}, \mathbf{H}, \mathbf{B})$ where X is a set of points, $\mathbf{G}$ and $\mathbf{H}$ are partitions of X (groups and holes, respectively) and $\mathbf{B}$ is a collection of subsets of X (blocks) such that

1. for each block $B \in \mathbf{B}$ and each $H \in \mathbf{H}$, $|B \cap H| \leq 1$, and
2. any pair of distinct points from X which are not in the same hole occur in some group or in exactly λ blocks, but not both.

A K-DGDD of type $(g, h^v a^1)^u$ is a double group divisible design in which every block has a size from the set K and in which there are u groups of size g, each of which intersects each of the first v holes in h points and the last hole in a points. Thus, $g = hv + a$. For example, a *k*-DGDD of type $(g, h^v a^1)^k$ is a *holey transversal design k*-HTD of hole type $h^v a^1$ and is equivalent to a set of $k-2$ holey MOLS of type $h^v a^1$.

Theorem 1.7 ([5], [19]). *There exists a* $4-\mathrm{DGDD}_\lambda(hv, h^v)^u$ *if, and only if,* $u, v \geq 4$ *and* $\lambda(u-1)(v-1)h \equiv 0 \pmod 3$ *except for* $(u, h, v, \lambda) = (4, 1, 6, 1)$.

Construction 1.8 ([18]). *Supposed that there is a* $4-\mathrm{DGDD}(gu, g^u)^n$, *and a 4-GDD of type* $g^u m^1$, $g > 1$, $u \geq 4$, *where* m *is a non-negative integer, then there exists a 4-GDD of type* $(n g)^u m^1$.

Since there also exists a 4-GDD of type 4^4, we obtain by Wilson's Fundamental Construction (WFC) [20]:

Corollary 1.9. *Supposed there exists a 4-GDD of type* $g^u m^1$, $g > 1$, $u \geq 4$, *then there exist a 4-GDD of type* $(4g)^u m^1$ *and a 4-GDD of type* $(4g)^u (4m)^1$.

Theorem 1.10 ([34]). *Supposed h and v are positive integers and a is non-negative, then there exists a 4-HTD of hole type* $h^v a^1$ *if, and only if,* $v \geq 4$ *and* $0 \leq a \leq h(v-1)/2$ *except for* $(h, v, a) = (1, 5, 1)$ *or* $(1, 6, 0)$.

Construction 1.11 [21], [13]). *Supposed there exists a* $4-$HTD *of hole type* $h^v a^1$, *then there exists a* $\{3,4\}-$DGDD *of type* $(3hv,(3h)^v)^4$ *whose blocks of size 3 can be partitioned into* $9a$ *parallel classes.*

Theorem 1.12 ([33]). *Let* m, n *be two positive integers. Then there exists a 4-GDD of type* $(3m)^4(6m)^1(3n)^1$ *if, and only if,* $m \le n \le 2m$ *with four possible exceptions* $(m, n) = (3,5), (4,7), (6,7)$, *or* $(6,11)$.

Construction 1.13 ([1]). *Suppose a* $\mathrm{TD}(k+1, n)$ *exists. Let* $\delta = 0$ *or* 1, *and form a block of size* $n+\delta$ *of each group together with* δ *infinite points. Now delete a finite point and use its blocks to define new groups. This results in a* $\{k+1, n+\delta\}-$GDD *of type* $k^n(n-1+\delta)^1$.

The following both constructions are extensively used throughout the paper.

Construction 1.14. *If there exists a 4-RGDD of type* g^u *then (by completing the* $g(u-1)/3$ *parallel classes with* $g(u-1)/3$ *new points) we get the existence of a 5-GDD of type* $g^u(g(u-1)/3)^1$ *whose blocks are all incident with the last group.*

The next construction is a variation of the WFC.

Construction 1.15. *If there exists a 5-GDD of type* $g^u l^1$ *whose blocks are all incident with the last group and there exists a 4-GDD of type* $x^4 a^1$ *for any* $a \equiv 0 \pmod{n}$ *and* $0 \le a \le nt$, *then there exists a 4-GDD of type* $(xg)^u m^1$ *for* $m \equiv 0 \pmod{n}$ *and* $0 \le m \le ntl$.

The concept of labeled resolvable designs is needed in order to get direct constructions for resolvable designs. This concept was introduced by Shen [28], [30], [31].

Let (X, B) be a $\mathrm{GDD}_\lambda(K, M; v)$ where $X = \{a_1, a_2, \ldots, a_v\}$ is totally ordered with ordering $a_1 < a_2 < \ldots < a_v$. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$ it is supposed that $x_1 < x_2 < \ldots < x_k$. Let Z_λ be the group of residues modulo λ.

Let $\varphi : \mathrm{B} \to Z_\lambda^{\binom{k}{2}}$ be a mapping where for each $B = \{x_1, x_2, \ldots, x_k\} \in \mathrm{B}$, $k \in K$, $\varphi(B) = (\varphi(x_1, x_2), \ldots, \varphi(x_1, x_k), \varphi(x_2, x_3), \ldots, \varphi(x_2, x_k), \varphi(x_3, x_4), \ldots, \varphi(x_{k-1}, x_k))$, $\varphi(x_i, x_j) \in Z_\lambda$ for $1 \le i < j \le k$.

A GDD$_\lambda(K, M; v)$ is said to be a *labeled group divisible design*, denoted by LGDD$_\lambda(K, M; v)$, if there exists a mapping φ such that:

1. For each pair $\{x, y\} \subset X$ with $x < y$, contained in the blocks $B_1, B_2, \ldots, B_\lambda$, then $\varphi_i(x,y) \equiv \varphi_j(x,y) \pmod{\lambda}$ if, and only if, $i = j$ where the subscripts i and j denote the blocks to which the pair belongs, for $1 \le i, j \le \lambda$; and
2. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, $\varphi(x_r, x_s) + \varphi(x_s, x_t) \equiv \varphi(x_r, x_t) \pmod{\lambda}$, for $1 \le r < s < t \le k$.

The blocks of φ will be denoted in the following form:

$(x_1\ \ x_2 \ldots x_k;\ \varphi(x_1,x_2) \ldots \varphi(x_1,x_k)\ \ \varphi(x_2,x_3) \ldots \varphi(x_2,x_k)\ \ \varphi(x_3,x_4) \ldots \varphi(x_{k-1},x_k))$ with $k \in K$.

The above definition is used for the first time in [24] and is a little bit more general than the definition by Shen [31] with $K = \{k\}$ or Shen and Wang [30] for transversal designs. The main application of the labeled designs is to blow up the point set of a given design with the following theorem (Shen, [28]) here extended for labeled pairwise balanced designs with some uniformly parallel classes.

Theorem 1.16 ([28], [24], [25]). *If there exists a* $K-$LGDD$_\lambda$ *of type* $g_1^{u_1} g_2^{u_2} \ldots g_s^{u_s}$ *with* r_k^L *parallel classes of size* k, *for each* $k \in K$, *then there exists a* $K-$GDD *of type* $(\lambda g_1)^{u_1} (\lambda g_2)^{u_2} \ldots (\lambda g_s)^{u_s}$ *with* $r_k = r_k^L$ *parallel classes of size* k, *for each* $k \in K$.

Theorem 1.17 ([26]). *All admissible 4-GDDs of type* $g^u m^1$ *exist for* $g \in \{8, 12, 16, 24, 48, 72, 96, 120, 144\}$.

In Section 2 we construct some new labeled designs which will be used as ingredients for our main recursive constructions in Section 3. Since the case of $u = 6$ is an exception in many recursive constructions in Section 3, all 4-GDDs of type $g^6 m^1$ with $g \equiv 0 \pmod{6}$ are determined in Section 4. In Section 5 all spectra of 4-GDDs of type $g^u m^1$ are constructed, where g is a multiple of 24 or 36.

2. Direct constructions

All following designs were found computationally.

Lemma 2.1. *There exists a 4-GDD of type* $10^6 m^1$ *for* $m \in \{4, 7, 13, 16, 19\}$.

Proof A $\{3,4\}-LGDD_{10}$ of type 1^6 with all blocks of size 3 partitioned into 4 3-pcs (each 3-pc is a row):
(2 3 6; 7 8 1), (1 4 5; 7 3 6),
(3 5 6; 0 2 2), (1 2 4; 9 6 7),
(1 2 5; 4 0 6), (3 4 6; 2 0 8),
(2 3 5; 6 4 8), (1 4 6; 5 9 4),
(1 2 4 6; 1 4 7 3 6 3), (1 2 5 6; 3 8 6 5 3 8), (2 4 5 6; 9 2 5 3 6 3),
(1 3 4 5; 3 1 6 8 3 5), (3 4 5 6; 0 9 5 9 5 6), (1 3 5 6; 5 1 8 6 3 7),
(2 3 4 5; 8 2 9 4 1 7), (2 4 5 6; 5 3 2 8 7 9), (1 4 5 6; 3 7 2 4 9 5),
(1 2 3 4; 7 1 8 4 1 7), (2 3 4 6; 5 4 4 9 9 0), (2 3 4 5; 9 0 1 1 2 1),
(1 2 3 6; 0 2 0 2 0 8), (1 2 3 4; 6 6 2 0 6 6), (1 3 4 6; 4 9 1 5 7 2),
(1 3 5 6; 0 4 4 4 4 0), (1 2 3 5; 5 8 5 3 0 7), (1 2 5 6; 2 9 3 7 1 4),
(1 2 3 6; 8 9 5 1 7 6), (2 4 5 6; 8 8 9 0 1 1), (1 3 4 5; 7 0 2 3 5 2).

A $\{3,4\}-LGDD_{10}$ of type 1^6 with all blocks of size 3 partitioned into 7 3-pcs (each 3-pc is a row):
(1 5 6; 0 2 2), (2 3 4; 4 7 3),
(2 3 5; 6 0 4), (1 4 6; 0 4 4),
(1 3 4; 4 8 4), (2 5 6; 3 6 3),
(2 3 6; 8 0 2), (1 4 5; 1 3 2),
(2 4 6; 2 8 6), (1 3 5; 9 9 0),
(1 2 5; 5 1 6), (3 4 6; 9 9 0),
(1 2 6; 2 1 9), (3 4 5; 5 2 7),
(1 2 4 5; 6 4 8 8 2 4), (2 3 4 5; 7 4 4 7 7 0), (1 2 3 4; 7 6 2 9 5 6),
(1 3 5 6; 8 7 8 9 0 1), (1 2 4 6; 8 7 9 9 1 2), (1 3 5 6; 1 6 5 5 4 9),
(1 3 4 6; 3 5 6 2 3 1), (1 2 3 4; 3 5 6 2 3 1), (2 4 5 6; 1 7 4 6 3 7),
(2 3 5 6; 5 8 3 3 8 5), (1 2 4 5; 9 9 4 0 5 5), (1 2 3 6; 1 2 3 1 2 1),
(1 2 3 6; 0 0 7 0 7 7), (3 4 5 6; 0 1 5 1 5 4), (2 4 5 6; 6 9 5 3 9 6),
(1 2 3 5; 4 7 5 3 1 8), (3 4 5 6; 8 6 6 8 8 0), (1 4 5 6; 3 2 0 9 7 8).

A $\{3,4\}-LGDD_{10}$ of type 1^6 with all blocks of size 3 partitioned into 13 3-pcs (each 3-pc is a row):
(2 4 6; 8 8 0), (1 3 5; 5 1 6),
(1 2 4; 3 6 3), (3 5 6; 7 1 4),
(2 4 5; 1 3 2), (1 3 6; 9 2 3),
(3 4 6; 9 5 6), (1 2 5; 1 8 7),
(1 2 3; 5 0 5), (4 5 6; 4 7 3),

(1 4 6; 0 4 4), (2 3 5; 2 1 9),
(1 4 5; 8 6 8), (2 3 6; 4 4 0),
(1 5 6; 7 5 8), (2 3 4; 8 2 4),
(2 5 6; 6 3 7), (1 3 4; 8 5 7),
(2 4 5; 6 9 3), (1 3 6; 2 1 9),
(1 5 6; 3 9 6), (2 3 4; 0 5 5),
(3 4 5; 0 0 0), (1 2 6; 2 8 6),
(4 5 6; 9 8 9), (1 2 3; 7 6 9),
(1 3 4 5; 3 4 5 1 2 1), (1 2 4 6; 4 1 6 7 2 5), (1 2 4 6; 8 2 3 4 5 1),
(1 3 4 6; 4 7 0 3 6 3), (1 2 4 5; 9 9 4 0 5 5), (2 3 4 6; 3 9 1 6 8 2),
(2 3 5 6; 6 0 0 4 4 0), (3 4 5 6; 8 5 7 7 9 2), (1 2 5 6; 0 2 7 2 7 5),
(1 2 3 5; 6 7 0 1 4 3), (1 3 4 5; 1 3 9 2 8 6), (2 3 5 6; 7 8 9 1 2 1).

A $\{3, 4\}-LGDD_{10}$ of type 1^6 with all blocks of size 3 partitioned into 16 3-pcs (each 3-pc is a row):
(1 3 6; 2 1 9), (2 4 5; 1 1 0),
(2 5 6; 6 4 8), (1 3 4; 1 0 9),
(1 5 6; 1 4 3), (2 3 4; 5 6 1),
(1 4 6; 7 0 3), (2 3 5; 0 9 9),
(1 3 5; 5 5 0), (2 4 6; 9 1 2),
(1 2 4; 5 5 0), (3 5 6; 4 5 1),
(1 5 6; 8 5 7), (2 3 4; 8 2 4),
(2 3 6; 7 7 0), (1 4 5; 8 6 8),
(2 5 6; 5 9 4), (1 3 4; 8 4 6),
(1 2 5; 1 4 3), (3 4 6; 7 2 5),
(1 3 5; 7 9 2), (2 4 6; 8 5 7),
(1 4 6; 2 8 6), (2 3 5; 2 0 8),
(3 4 5; 5 1 6), (1 2 6; 4 7 3),
(1 2 6; 0 6 6), (3 4 5; 0 7 7),
(4 5 6; 5 0 5), (1 2 3; 2 6 4),
(1 2 3; 7 0 3), (4 5 6; 2 1 9),
(1 2 3 6; 3 9 3 6 0 4), (2 3 5 6; 9 2 2 3 3 0), (3 4 5 6; 2 5 1 3 9 6),
(1 2 4 5; 9 6 7 7 8 1), (1 2 4 5; 6 9 3 3 7 4), (1 2 4 5; 8 3 2 5 4 9),
(2 3 4 6; 1 4 8 3 7 4), (1 3 4 6; 3 1 9 8 6 8), (1 3 5 6; 4 0 2 6 8 2).

A $\{3, 4\}-LGDD_{10}$ of type 1^6 with all blocks of size 3 partitioned into 19 3-pcs (each 3-pc is a row):
(3 4 5; 5 3 8), (1 2 6; 1 4 3),
(3 4 6; 4 1 7), (1 2 5; 3 9 6),
(1 2 5; 2 6 4), (3 4 6; 3 6 3),
(1 5 6; 7 8 1), (2 3 4; 9 9 0),
(1 4 6; 9 5 6), (2 3 5; 8 5 7),
(1 3 6; 2 6 4), (2 4 5; 8 9 1),

(3 5 6; 9 5 6), (1 2 4; 4 6 2),
(1 5 6; 1 3 2), (2 3 4; 4 6 2),
(1 3 5; 6 0 4), (2 4 6; 5 4 9),
(1 4 5; 2 4 2), (2 3 6; 1 9 8),
(1 3 4; 8 4 6), (2 5 6; 7 0 3),
(2 4 5; 7 3 6), (1 3 6; 9 1 2),
(1 2 3; 6 3 7), (4 5 6; 0 0 0),
(2 4 6; 3 7 4), (1 3 5; 1 3 2),
(1 2 6; 8 9 1), (3 4 5; 8 1 3),
(1 2 4; 0 1 1), (3 5 6; 5 9 4),
(1 4 6; 0 2 2), (2 3 5; 6 2 6),
(2 3 6; 2 2 0), (1 4 5; 3 8 5),
(1 3 4; 7 8 1), (2 5 6; 1 6 5),
(1 2 3 6; 9 4 7 5 8 3), (1 2 3 4; 7 0 7 3 0 7), (3 4 5 6; 9 8 7 9 8 9),
(1 4 5 6; 5 2 0 7 5 8), (1 2 3 5; 5 5 5 0 0 0), (2 4 5 6; 4 8 5 4 1 7).

The assertions comes from Theorem 1.16 by completing all 3-pcs. □

Lemma 2.2. *There exists a 4-GDD of type* $10^6 22^1$.

Proof A $\{3,4\}-\text{LGDD}_5$ of type 2^6 with all blocks of size 3 partitioned into 22 3-pcs (each 3-pc is a row); $G=\{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\},\{11,12\}\}$:
(2 4 9; 0 0 0), (1 3 7; 3 2 4), (6 10 11; 0 1 1), (5 8 12; 1 3 2),
(3 5 11; 2 0 3), (1 8 9; 0 0 0), (4 6 7; 3 2 4), (2 10 12; 0 0 0),
(3 10 12; 0 2 2), (1 8 11; 3 1 3), (2 6 9; 0 4 4), (4 5 7; 3 0 2),
(1 7 12; 1 3 2), (3 5 9; 0 3 3), (6 8 10; 1 4 3), (2 4 11; 2 1 4),
(3 9 12; 0 3 3), (4 5 7; 0 3 3), (1 6 10; 3 1 3), (2 8 11; 2 4 2),
(1 9 12; 4 0 1), (4 8 10; 3 0 2), (2 5 7; 4 4 0), (3 6 11; 0 2 2),
(5 7 9; 4 2 3), (1 6 8; 0 4 4), (4 10 12; 1 4 3), (2 3 11; 4 3 4),
(3 6 9; 2 2 0), (7 10 12; 2 3 1), (2 5 8; 2 1 4), (1 4 11; 3 3 0),
(1 4 10; 0 2 2), (2 7 11; 2 0 3), (5 9 12; 0 2 2), (3 6 8; 4 2 3),
(1 3 10; 2 4 2), (7 9 11; 2 4 2), (2 6 12; 3 4 1), (4 5 8; 4 4 0),
(3 6 7; 1 1 0), (1 4 12; 1 4 3), (5 10 11; 3 1 3), (2 8 9; 4 1 2),
(6 8 12; 2 2 0), (2 3 7; 3 3 0), (1 4 10; 2 0 3), (5 9 11; 4 0 1),
(1 8 9; 2 3 1), (7 10 12; 0 4 4), (2 4 6; 3 2 4), (3 5 11; 4 1 2),
(2 5 12; 3 3 0), (4 6 10; 2 4 2), (3 7 9; 2 1 4), (1 8 11; 1 2 1),
(2 3 10; 0 3 3), (1 5 11; 0 4 4), (7 9 12; 1 0 4), (4 6 8; 1 1 0),
(2 6 12; 4 2 3), (8 10 11; 0 0 0), (1 3 5; 4 2 3), (4 7 9; 4 4 0),
(2 7 10; 0 4 4), (4 8 12; 2 1 4), (6 9 11; 1 4 3), (1 3 5; 0 1 1),
(1 6 9; 4 2 3), (7 10 11; 3 0 2), (2 3 8; 2 3 1), (4 5 12; 1 2 1),
(5 8 10; 2 1 4), (1 6 7; 1 3 2), (3 9 12; 4 4 0), (2 4 11; 4 2 3),
(2 4 9; 1 2 1), (6 7 11; 1 3 2), (1 5 10; 4 3 4), (3 8 12; 3 1 3),

(1 5 12; 3 2 4), (2 6 9; 1 3 2), (4 7 11; 1 2 1), (3 8 10; 0 1 1),
(2 5 10; 1 1 0), (1 3 7; 1 4 3), (4 6 12; 0 0 0), (8 9 11; 4 4 0),
(2 3 8 12; 1 0 1 4 0 1), (3 6 10 11; 3 4 3 1 0 4), (4 5 8 9; 2 0 3 3 1 3),
(1 4 9 11; 4 1 0 2 1 4), (2 5 7 10; 0 1 2 1 2 1), (1 6 7 12; 2 0 1 3 4 1).

The assertion comes from Theorem 1.16 by completing all 3-pcs. □

Lemma 2.3. *There exists a* $\{3,4\}-\mathrm{LGDD}_{18}$ *of type* 1^6 *with all blocks of size 3 partitioned into* m *3-pcs for* $m \in \{3, 6, 12, 15\}$.

Proof A $\{3,4\}-\mathrm{LGDD}_{18}$ of type 1^6 with all blocks of size 3 partitioned into 3 3-pcs (each 3-pc is a row):
(1 3 6; 13 9 14), (2 4 5; 2 6 4),
(4 5 6; 16 9 11), (1 2 3; 6 16 10),
(1 4 5; 10 15 5), (2 3 6; 2 17 15),
(2 3 4 5; 6 14 8 8 2 12), (1 2 5 6; 0 17 0 17 0 1), (1 3 4 6; 1 4 4 3 3 0),
(1 3 4 5; 4 6 9 2 5 3), (1 2 3 5; 16 5 1 7 3 14), (3 4 5 6; 14 9 16 13 2 7),
(3 4 5 6; 4 13 7 9 3 12), (2 3 4 6; 14 9 16 13 2 7), (2 3 5 6; 8 11 3 3 13 10),
(2 3 5 6; 3 0 9 15 6 9), (1 3 5 6; 10 4 10 12 0 6), (1 3 4 6; 9 7 3 16 12 14),
(2 4 5 6; 11 10 6 17 13 14), (1 2 5 6; 13 0 2 5 7 2), (1 4 5 6; 2 3 8 1 6 5),
(1 2 5 6; 12 8 6 14 12 16), (1 3 4 5; 6 17 5 11 17 6), (1 2 4 6; 15 0 5 3 8 5),
(2 3 4 6; 4 1 13 15 9 12), (1 2 3 5; 10 3 7 11 15 4), (1 2 4 6; 3 15 7 12 4 10),
(2 4 5 6; 5 12 2 7 15 8), (1 2 3 6; 11 8 12 15 1 4), (1 2 4 6; 14 12 11 16 15 17),
(1 2 4 5; 7 14 11 7 4 15), (3 4 5 6; 7 7 11 0 4 4), (1 3 4 6; 11 11 1 0 8 8),
(2 3 5 6; 13 1 14 6 1 13), (1 2 3 4; 4 2 1 16 15 17), (1 3 5 6; 12 2 17 8 5 15),
(1 3 4 5; 15 3 13 6 16 10), (2 3 4 5; 12 17 13 5 1 14), (1 4 5 6; 5 16 16 11 11 0),
(1 3 5 6; 14 14 13 0 17 17), (2 3 4 6; 1 13 11 12 10 16),
(1 2 3 4; 17 17 9 0 10 10), (1 2 3 4; 2 7 8 5 6 1), (2 3 4 5; 9 0 2 9 11 2),
(1 2 4 5; 8 16 6 8 16 8), (1 2 3 5; 1 0 10 17 9 10), (1 2 4 6; 9 13 14 4 5 1),
(1 2 5 6; 5 12 15 7 10 3).

A $\{3,4\}-\mathrm{LGDD}_{18}$ of type 1^6 with all blocks of size 3 partitioned into 6 3-pcs (each 3-pc is a row):
(1 3 5; 12 11 17), (2 4 6; 7 14 7),
(1 3 6; 15 14 17), (2 4 5; 11 17 6),
(1 2 3; 10 3 11), (4 5 6; 15 0 3),
(1 4 5; 7 2 13), (2 3 6; 2 8 6),
(1 4 6; 2 11 9), (2 3 5; 17 9 10),
(1 2 6; 2 17 15), (3 4 5; 6 7 1),
(1 2 3 4; 6 9 8 3 2 17), (1 2 5 6; 12 10 15 16 3 5), (3 4 5 6; 10 0 8 8 16 8),
(3 4 5 6; 14 16 2 2 6 4), (1 3 4 5; 6 0 17 12 11 17), (1 3 4 5; 11 16 8 5 15 10),
(2 4 5 6; 4 13 6 9 2 11), (1 2 3 4; 5 17 3 12 16 4), (3 4 5 6; 15 4 16 7 1 12),
(1 2 3 4; 15 2 15 5 0 13), (2 3 4 6; 1 8 16 7 15 8), (1 2 4 5; 0 12 6 12 6 12),

(1 4 5 6; 5 5 0 0 13 13), (2 3 4 5; 10 1 12 9 2 11), (2 3 5 6; 14 8 0 12 4 10),
(1 2 4 6; 16 11 3 13 5 10), (1 3 4 6; 16 9 12 11 14 3), (1 2 4 5; 7 10 14 3 7 4),
(2 3 5 6; 0 14 13 14 13 17), (1 2 5 6; 14 0 6 4 10 6), (1 3 5 6; 4 7 4 3 0 15),
(1 4 5 6; 6 9 10 3 4 1), (2 3 4 6; 8 10 9 2 1 17), (1 3 5 6; 13 4 2 9 7 16),
(2 3 4 6; 7 15 12 8 5 15), (1 2 4 5; 13 1 15 6 2 14), (1 2 5 6; 8 1 1 11 11 0),
(2 3 5 6; 9 10 1 1 10 9), (1 2 4 6; 17 4 16 5 17 12), (1 2 3 5; 9 7 12 16 3 5),
(1 2 3 4; 4 10 13 6 9 3), (1 3 4 6; 1 17 13 16 12 14), (2 3 4 5; 13 14 1 1 6 5),
(1 2 3 6; 1 5 8 4 7 3), (1 2 5 6; 3 3 5 0 2 2), (1 3 5 6; 0 13 9 13 9 14),
(1 2 3 5; 11 8 16 15 5 8), (2 4 5 6; 17 15 4 16 5 7), (1 3 4 6; 14 14 7 0 11 11).

A $\{3,4\}-LGDD_{18}$ of type 1^6 with all blocks of size 3 partitioned into 12 3-pcs (each 3-pc is a row):
(2 3 4; 14 7 11), (1 5 6; 13 16 3),
(1 2 4; 4 2 16), (3 5 6; 4 8 4),
(2 4 6; 2 9 7), (1 3 5; 6 17 11),
(1 4 5; 10 9 17), (2 3 6; 10 5 13),
(1 2 3; 8 17 9), (4 5 6; 10 12 2),
(3 4 6; 3 6 3), (1 2 5; 1 4 3),
(1 3 4; 4 1 15), (2 5 6; 11 3 10),
(2 3 5; 2 16 14), (1 4 6; 17 0 1),
(1 3 4; 13 8 13), (2 5 6; 8 7 17),
(1 3 5; 8 7 17), (2 4 6; 8 14 6),
(2 3 4; 17 11 12), (1 5 6; 12 7 13),
(1 4 6; 16 6 8), (2 3 5; 16 6 8),
(1 3 4 5; 3 7 0 4 15 11), (1 2 3 6; 17 12 17 13 0 5), (1 2 4 5; 12 9 16 15 4 7),
(1 3 5 6; 2 8 1 6 17 11), (2 4 5 6; 3 1 1 16 16 0), (2 4 5 6; 14 2 11 6 15 9),
(2 3 4 5; 12 1 10 7 16 9), (1 3 4 6; 16 12 5 14 7 11), (1 2 3 6; 3 11 11 8 8 0),
(1 2 4 6; 10 4 8 12 16 4), (1 2 4 6; 0 5 4 5 4 17), (1 2 5 6; 16 3 10 5 12 7),
(1 2 4 5; 7 6 2 17 13 14), (1 2 5 6; 6 15 3 9 15 6), (1 2 3 4; 5 9 14 4 9 5),
(2 3 4 5; 0 0 12 0 12 12), (2 3 4 6; 15 6 6 9 9 0), (1 2 3 6; 14 7 9 11 13 2),
(2 3 5 6; 7 14 10 7 3 14), (2 3 4 5; 5 4 7 17 2 3), (1 2 3 5; 9 15 6 6 15 9),
(3 4 5 6; 16 0 12 2 14 12), (3 4 5 6; 10 10 15 0 5 5), (1 4 5 6; 0 5 13 5 13 8),
(3 4 5 6; 6 1 16 13 10 15), (3 4 5 6; 2 3 4 1 2 1), (1 2 4 5; 11 3 11 10 0 8),
(1 3 4 6; 10 11 2 1 10 9), (1 3 5 6; 1 14 12 13 11 16), (1 2 4 5; 2 15 1 13 17 4),
(1 2 3 6; 15 0 14 3 17 14), (1 3 4 5; 5 13 10 8 5 15), (1 2 3 6; 13 14 15 1 2 1).

A $\{3,4\}-LGDD_{18}$ of type 1^6 with all blocks of size 3 partitioned into 15 3-pcs (each 3-pc is a row):
(3 4 6; 12 10 16), (1 2 5; 4 15 11),
(4 5 6; 0 5 5), (1 2 3; 0 6 6),
(1 4 6; 17 10 11), (2 3 5; 3 9 6),
(2 3 6; 4 7 3), (1 4 5; 15 17 2),
(1 2 3; 17 12 13), (4 5 6; 17 14 15),

(3 5 6; 3 13 10), (1 2 4; 7 9 2),
(1 3 6; 5 17 12), (2 4 5; 16 10 12),
(3 5 6; 5 5 0), (1 2 4; 1 8 7),
(2 4 6; 15 6 9), (1 3 5; 14 4 8),
(2 4 6; 17 16 17), (1 3 5; 7 9 2),
(1 3 6; 2 9 7), (2 4 5; 13 8 13),
(2 3 4; 15 14 17), (1 5 6; 5 11 6),
(1 4 6; 16 1 3), (2 3 5; 7 1 12),
(1 3 4; 0 6 6), (2 5 6; 16 14 16),
(2 4 6; 1 2 1), (1 3 5; 10 14 4),
(1 2 5 6; 9 8 12 17 3 4), (1 2 3 6; 6 4 5 16 17 1), (2 3 4 5; 2 4 2 2 0 16),
(1 4 5 6; 0 3 2 3 2 17), (1 2 4 5; 12 4 12 10 0 8), (1 2 5 6; 13 16 6 3 11 8),
(1 2 4 6; 14 1 8 5 12 7), (3 4 5 6; 8 15 16 7 8 1), (1 2 3 6; 16 8 7 10 9 17),
(1 4 5 6; 7 11 13 4 6 2), (1 2 3 4; 3 15 11 12 8 14), (1 3 4 6; 11 12 4 1 11 10),
(2 3 5 6; 14 6 0 10 4 12), (1 2 4 5; 5 14 1 9 14 5), (1 3 4 5; 17 3 13 4 14 10),
(1 3 4 6; 16 5 0 7 2 13), (2 3 4 6; 9 0 15 9 6 15), (1 3 4 5; 13 10 6 15 11 14),
(2 3 5 6; 0 13 8 13 8 13), (1 2 5 6; 11 0 3 7 10 3), (2 3 4 6; 8 11 5 3 15 12),
(1 2 3 4; 8 9 2 1 12 11), (1 2 3 6; 2 1 15 17 13 14), (3 4 5 6; 5 16 9 11 4 11),
(1 3 4 5; 3 13 10 10 7 15), (1 2 5 6; 15 2 16 5 1 14), (2 3 4 5; 11 6 12 13 1 6),
(1 2 5 6; 10 7 14 15 4 7), (2 3 4 5; 5 3 4 16 17 1), (3 4 5 6; 0 9 0 9 0 9). □

Lemma 2.4. *There exists a 4-GDD of type* $18^6 m^1$ *for* $m \in \{24, 30, 33, 39, 42\}$..

Proof A $\{3, 4\}-\text{LGDD}_9$ of type 2^6 with all blocks of size 3 partitioned into m 3-pcs, $G = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}, \{11,12\}\}$, $m \in \{24, 30, 33, 39, 42\}$, is applied as Online Resource [35] Designs 1-5, which results in a {3, 4}-GDD of type 18^6 with all blocks of size 3 partitioned into m 3-pcs for $m \in \{24, 30, 33, 39, 42\}$ by Theorem 1.16. By completing all 3-pcs, we obtain the desired designs. □

Lemma 2.5. *There exists a 4-GDD of type* $90^6 m^1$ *for* $m \in \{210, 213, 219, 222\}$.

Proof A $\{3, 4\}-\text{LGDD}_{30}$ of type 3^6 with all blocks of size 3 partitioned into m 3-pcs, $m \in \{210, 213, 219, 222\}$, is provided as Online Resource [35] Designs 6-9, which results in a {3, 4}-GDD of type 90^6 with all blocks of size 3 partitioned into m 3-pcs by Theorem 1.16. By completing all 3-pcs, we obtain the desired designs. □

Lemma 2.6. *There exists a 4-GDD of type* $60^6 m^1$ *for* $m \in \{141, 144, 147\}$.

Proof A $\{3,4\}-LGDD_{60}$ of type 1^6 with all blocks of size 3 partitioned into m 3-pcs, $m\in\{141, 144\}$, is provided as Online Resource [35] Designs 10-11, which results in a {3, 4}-GDD of type 60^6 with all blocks of size 3 partitioned into m 3-pcs by Theorem 1.16. By completing all 3-pcs, we obtain the desired designs for $m\in\{141, 144\}$.

A $\{3,4\}-LGDD_{30}$ of type 2^6 with all blocks of size 3 partitioned into 147 3-pcs, $G=\{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\},\{11,12\}\}$, is given as Online Resource [35] Design 12, which results in a {3, 4}-GDD of type 60^6 with all blocks of size 3 partitioned into 147 3-pcs by Theorem 1.16. By completing all 3-pcs, we obtain a 4-GDD of type 60^6147^1. □

Lemma 2.7. *There exists a 4-GDD of type* 10^9m^1 *for* $m\in\{7,13,16,19,22,25,28,31,34,37\}$.

Proof There exists a $\{3,4\}-LGDD_{10}$ of type 1^9 with all blocks of size 3 partitioned into m 3-pcs, $m\in\{16,22,28,34\}$, by the Online Resource [35] Designs 13-16. There exists a $\{3,4\}-LGDD_5$ of type 2^9 with all blocks of size 3 partitioned into m 3-pcs, $m\in\{7,13,19,25,31,37\}$, by the Online Resource [35] Designs 17-22. The assertion follows by Theorem 1.16 and by completing all 3-pcs. □

Lemmas 2.8 -2.13 show the existence of {3, 4}-GDDs of type 10^u with all blocks of size 3 in 3-pcs for small u.

Lemma 2.8. *There exists a* {3, 4}-GDD *of type* 10^9 *with all blocks of size* 3 *in 4 3-pcs.*

Proof Points: $\mathbb{Z}_{90}$. Groups: $\{\{i,i+9,i+18,\dots,i+81\}, i=0,1,\dots,8\}$.
Blocks: Develop the following base blocks +3 (mod 90), {0, 22, 35}, {1, 23, 36}, {2, 24, 37}, to obtain 3 3-pcs. Develop the following short base block +1 (mod 90); {0, 30, 60} to obtain a 3-pc.
Develop the following base blocks +1 (mod 90): {0, 2, 28, 44}, {0, 4, 37, 43}, {0, 5, 19, 29}, {0, 7, 38, 41}, {0, 15, 23, 40}, {0, 20, 21, 32}. □

Lemma 2.9 *There exists a* {3, 4}-GDD *of type* 10^{12} *with all blocks of size* 3 *in 7 3-pcs*

Proof Points: $\mathbb{Z}_{120}$. Groups: $\{\{i,i+12,i+24,\dots,i+108\}, i=0,1,\dots,11\}$.

Blocks: Develop the following base blocks +3 (mod 120), {0, 1, 44}, {1, 2, 45}, {2, 3, 46}, {0, 34, 56}, {1, 35, 57}, {2, 36, 58} to obtain 6 3-pcs. Develop the following short base block +1 (mod 120), {0, 40, 80} to obtain a 3-pc.
Develop the following base blocks +1 (mod 120): {0, 2, 37, 53}, {0, 3, 7, 57}, {0, 5, 33, 46}, {0, 8, 38, 55}, {0, 9, 27, 58}, {0, 10, 29, 52}, {0, 11, 26, 32}, {0, 14, 39, 59}. □

Lemma 2.10. *There exists a* {3, 4}-GDD *of type* 10^{15} *with all blocks of size* 3 *in* 4 *3-pcs.*

Proof Points: $\mathbb{Z}_{150}$. Groups: $\{\{i, i+15, i+30, ..., i+135\}, i=0,1,...,14\}$.
Blocks: Develop the following base blocks +3 (mod 150), {0, 25, 62}, {1, 26, 63}, {2, 27, 64} to obtain 3 3-pcs. Develop the following short base block +1 (mod 150), {0, 50, 100} to obtain a 3-pc.
Develop the following base blocks +1 (mod 150): {0, 6, 24, 64}, {0, 8, 46, 55}, {0, 13, 35, 69}, {0, 14, 42, 71}, {0, 17, 21, 65}, {0, 20, 72, 73}, {0, 23, 33, 59}, {0, 27, 66, 68}, {0, 31, 63, 74}, {0, 49, 54, 61}, {0, 51, 67, 70}. □

Lemma 2.11 *There exists a* {3, 4}-GDD *of type* 10^{18} *with all blocks of size* 3 *in* 7 *3-pcs*

Proof Points: $\mathbb{Z}_{180}$. Groups: $\{\{i, i+18\ i+36\ ..., i+162\}, i=0,1,...,17\}$.
Blocks: Develop the following base blocks +3 (mod 180), {0, 1, 80}, {1, 2, 81}, {2, 3, 82}, {0, 46, 86}, {1, 47, 87}, {2, 48, 88} to obtain 6 3-pcs. Develop the following short base block +1 (mod 180), {0, 60, 120} to obtain a 3-pc.
Develop the following base blocks +1 (mod 180): {0, 2, 70, 77}, {0, 3, 48, 87}, {0, 4, 33, 89}, {0, 8, 65, 81}, {0, 14, 42, 55}, {0, 15, 38, 82}, {0, 17, 49, 76}, {0, 19, 43, 53}, {0, 20, 50, 71}, {0, 22, 74, 83}, {0, 26, 63, 88}, {0, 31, 66, 78}, {0, 58, 64, 69}. □

Lemma 2.12. *There exists a* {3, 4}-GDD *of type* 10^{21} *with all blocks of size* 3 *in* 4 *3-pcs*

Proof Points: $\mathbb{Z}_{210}$. Groups: {0, 21, 42, ... , 189} (mod 210).
Blocks: Develop the following base blocks +3 (mod 210), {0, 82, 101}, {1, 83, 102}, {2, 84, 103} to obtain 3 3-pcs. Develop the following short base block +1 (mod 210), {0, 70, 140} to obtain a 3-pc. Develop the following base blocks +1 (mod 210): {0, 1, 47, 100}, {0, 2, 78, 95}, {0, 3, 12, 51}, {0, 5, 23, 77}, {0, 10, 74, 90}, {0, 11, 55, 92}, {0, 13, 35, 102}, {0, 20, 50, 86}, {0, 24, 62, 103}, {0, 26, 32, 91}, {0, 27, 88, 96}, {0, 28, 71, 85}, {0, 29, 97, 104}, {0, 31, 83, 87}, {0, 34, 49, 94}, {0, 40, 73, 98}, {0, 82, 101}. □

Lemma 2.13. *There exists a* $\{3, 4\}$-GDD *of type* 10^{27} *with all blocks of size* 3 *in* 4 *3-pcs*

Proof Points: $\mathbb{Z}_{270}$. Groups: $\{\{i, i+21\, i+42 \ldots, i+189\}, i=0,1,\ldots,20\}$.
Blocks: Develop the following base blocks +3 (mod 270); {0, 7, 104}, {1, 8, 105}, {2, 9, 106} to obtain 3 3-pcs. Develop the following short base block +1 (mod 270); {0, 90, 180} to obtain a 3-pc. Develop the following base blocks +1 (mod 270): {0, 2, 116, 122}, {0, 3, 89, 133}, {0, 5, 85, 103}, {0, 8, 75, 107}, {0, 15, 110, 132}, {0, 19, 88, 119}, {0, 20, 76, 125}, {0, 28, 73, 134}, {0, 30, 78, 113}, {0, 34, 94, 127}, {0, 36, 115, 128}, {0, 37, 87, 111}, {0, 38, 52, 129}, {0, 40, 41, 66}, {0, 42, 65, 124}, {0, 47, 63, 131}, {0, 51, 109, 121}, {0, 53, 57, 96}, {0, 55, 64, 126}, {0, 72, 101, 118}, {0, 102, 112, 123}. □

Lemma 2.14. *There exist 4-GDDs of types* $10^9 4^1$, $10^{12}7^1$, $10^{15}4^1$, $10^{18}7^1$, $10^{21}4^1$, $10^{27}4^1$.

Proof The assertions follow by Lemmas 2.8 - 2.13 by completing all 3-pcs. □.

Lemma 2.15. *There exists a 4-GDD of type* $20^6 m^1$ *for* $m \in \{8, 11, 14, 17, 23, 26, 29, 32, 38, 41, 44, 47\}$.

Proof A $\{3,4\}$-LGDD$_{20}$ of type 1^6 with all blocks of size 3 partitioned into m 3-pcs, $m \in \{8, 11, 14, 17, 23, 26, 29, 32, 38, 41, 44\}$, is provided as Online Resource [35] Designs 23-33, which results in a $\{3, 4\}$-GDD of type 20^6 with all blocks of size 3 partitioned into m 3-pcs by Theorem 1.16.
A $\{3,4\}$-LGDD$_{10}$ of type 2^6 with all blocks of size 3 partitioned into 47 3-pcs, is provided as Online Resource [35] Design 34, which results in a $\{3, 4\}$-GDD of type 20^6 with all blocks of size 3 partitioned into 47 3-pcs by Theorem 1.16.
By completing all 3-pcs, we obtain the desired designs. □

Lemma 2.16. *There exists a 4-GDD of type* $20^9 m^1$ *for* $m \in \{26, 38, 44, 62, 68, 74\} \cup \{29, 41, 47, 53, 59, 71, 77\}$.

Proof A $\{3,4\}$-LGDD$_{20}$ of type 1^9 with all blocks of size 3 partitioned into m 3-pcs, $m \in \{26, 38, 44, 62, 68, 74\}$, is provided as Online Resource [35] Designs 35-40, which results in a $\{3, 4\}$-GDD of type 20^6 with all blocks of size 3 partitioned into m 3-pcs by Theorem 1.16.

A $\{3,4\}$-LGDD$_{10}$ of type 2^9 with all blocks of size 3 partitioned into m 3-pcs, $m \in \{29, 41, 47, 53, 59, 71, 77\}$, is provided as Online Resource [35] Designs 41-47, which results in a $\{3, 4\}$-GDD of type 20^6 with all blocks of size 3 partitioned into m 3-pcs by Theorem 1.16. By completing all 3-pcs, we obtain the desired designs. □

Lemma 2.17. *There exists a 4-GDD of type $28^9 m^1$ for* $m \in \{34, 46, 52, 58, 76, 82, 94, 100, 106\}$

$$\cup \{37, 43, 55, 61, 67, 73, 79, 85, 97, 103, 109\}.$$

Proof A $\{3,4\}$-LGDD$_{28}$ of type 1^9 with all blocks of size 3 partitioned into m 3-pcs, $m \in \{34, 46, 52, 58, 76, 82, 94, 100, 106\}$, is provided as Online Resource [35] Designs 48-56, which results in a $\{3, 4\}$-GDD of type 28^9 with all blocks of size 3 partitioned into m 3-pcs by Theorem 1.16.
A $\{3,4\}$-LGDD$_{14}$ of type 2^9 with all blocks of size 3 partitioned into m 3-pcs, $m \in \{37, 43, 55, 61, 67, 73, 79, 85, 97, 103, 109\}$, is provided as Online Resource [35] Designs 57-67, which results in a $\{3, 4\}$-GDD of type 28^9 with all blocks of size 3 partitioned into m 3-pcs by Theorem 1.16. By completing all 3-pcs, we obtain the desired designs. □

Lemma 2.18. *There exists a 4-GDD of type $7^9 m^1$ for $m \in \{4, 10, 16, 22, 28\}$.*

Proof There exists a 4-GDD of type $7^9 m^1$ for $m \in \{4, 28\}$ by Theorem 1.2. A $\{3,4\}$-LGDD$_7$ of type 1^9 with all blocks of size 3 partitioned into m 3-pcs, $m \in \{10, 16, 22\}$, is provided as Online Resource [35] Designs 68-70, which results in a $\{3, 4\}$-GDD of type 7^9 with all blocks of size 3 partitioned into m 3-pcs by Theorem 1.16. By completing all 3-pcs, we obtain the desired designs. □

Lemma 2.19. *There exists a 4-GDD of type $18^5 m^1$ for $m \in \{12, 15, 21\}$.*

Proof A $\{3,4\}$-LGDD$_3$ of type 6^5 with all blocks of size 3 partitioned into m 3-pcs, $m \in \{12, 15, 21\}$, is provided as Online Resource [35] Designs 71-73, which results in a $\{3, 4\}$-GDD of type 18^5 with all blocks of size 3 partitioned into m 3-pcs by Theorem 1.16. By completing all 3-pcs, we obtain the desired designs. □

3. Recursive constructions

Theorem 3.1. *Let* $g \equiv 0 \pmod{72}$. *Then there exists a 4-GDD of type* $g^u m^1$ *if, and only if,* $u \geq 4$, $m \equiv 0 \pmod{3}$ *with* $0 \leq m \leq g(u-1)/2$.

Proof A 4-GDD of type $72^u m^1$ exists if, and only if, $u \geq 4$, $m \equiv 0 \pmod{3}$ and $0 \leq m \leq 36(u-1)$ by Theorem 1.17.
Therefore, let $g = 72n$, $n \geq 2$. There exists a 4-RGDD of type 12^u, $u \geq 4$ by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $12^u(4(u-1))^1$ which is our master design. There exists a 4-GDD of type $(6n)^4 a^1$, $a \equiv 0 \pmod{3}$, $0 \leq a \leq 9n$ for $n \geq 2$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights like a. All other points obtain weight $6n$. The result is a 4-GDD of type $(72n)^u m^1$, $m \equiv 0 \pmod{3}$ and $0 \leq m \leq 36n(u-1)$, $n \geq 2$, $u \geq 4$. □

Theorem 3.2. *If* $g \equiv 36 \pmod{72}$ *and for* $u \geq 4$, $u \neq 6$ *there exist all 4-GDDs of type* $g^u m_0^{\ 1}$, $m_0 \equiv 0 \pmod{3}$ *and* $0 \leq m_0 \leq g(u-1)/2$, *then there exist all 4-GDDs of type* $(3g)^u m^1$, $m \equiv 0 \pmod{3}$ *and* $0 \leq m \leq 3g(u-1)/2$.

Proof Let $g = 12n$, $n \equiv 3 \pmod{6}$. There exists a 4-RGDD of type 12^u, $u \geq 4$ by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $12^u(4(u-1))^1$ as our master design. There exists a 4-GDD of type $(3n)^4 a^1$, $a \equiv 0 \pmod{3}$, $0 \leq a \leq (9n-3)/2$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights like a. All other points obtain weight $3n$. The result is a 4-GDD of type $(36n)^u m^1$, $m \equiv 0 \pmod{3}$ and $0 \leq m \leq (18n-6)(u-1)$, $u \geq 4$.
There exists a TD$(4,u)$ for $u \geq 4$, $u \neq 6$ by Theorem 1.5. Remove a point and use this point to redefine the groups. Complete the groups of size u with a new point. This results in a $\{4, u+1\}$- GDD of type $3^u u^1$ as master design. There exists a 4-GDD of type $(12n)^4$ by Theorem 1.1 and 4-GDDs of type $(12n)^u a_0^{\ 1}$, $a_0 \equiv 0 \pmod{3}$, $0 \leq a_0 \leq 6n(u-1)$ by above premise. Every point in a group of size 3 in the master design is given weight $12n$. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain weight $12n$ and the new point obtains weights like a_0. The result is a 4-GDD of type $(36n)^u m^1$ with $m \equiv 0 \pmod{3}$, $12n(u-1) \leq m \leq 12n(u-1) + 6n(u-1) = 18n(u-1)$ for $u \geq 4$ and $u \neq 6$. □

Theorem 3.3. *If* $g \equiv 0 \pmod 6$ *and for* $u \geq 4$ *there exist all 4-GDDs of type* $g^u m_0^{\ 1}$, $m_0 \equiv 0 \pmod 3$ *and* $0 \leq m_0 \leq g(u-1)/2$, *then there exist all 4-GDDs of type* $(4g)^u m^1$ *for* $m \equiv 0 \pmod 3$ *and* $0 \leq m \leq 2g(u-1)$.

Proof For $h = g/3 \equiv 0 \pmod 2$ there exists a 4-HTD of hole type $h^v a^1$, $v \geq 4$ and $0 \leq a \leq h(v-1)/2$ by Theorem 1.10. Therefore, there exists a $\{3,4\}-$DGDD of type $(3hv,(3h)^v)^4$ whose blocks of size 3 can be partitioned into $9a$ parallel classes by Construction 1.11. Adjoin $9a$ infinite points to complete the 3-pcs and then adjoin further m_0 ideal points, filling in 4-GDDs of type $(3h)^v m_0^{\ 1}$, $m_0 \equiv 0 \pmod 3$, $0 \leq m_0 \leq 3h(v-1)/2$ coming from the premise to obtain a 4-GDD of type $(12h)^v m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 9h(v-1)/2 + 3h(v-1)/2 = 6h(v-1)$. Each value m can be combined because $3h(v-1)/2 \geq 9$. □

Theorem 3.4. *If* $g \equiv 36 \pmod{72}$ *and for* $u \geq 4$ *there exist all 4-GDDs of type* $g^u m_0^{\ 1}$, $m_0 \equiv 0 \pmod 3$ *and* $0 \leq m_0 \leq g(u-1)/2$, *then there exist all 4-GDDs of type* $(5g)^u m^1$, $m \equiv 0 \pmod 3$ *and* $0 \leq m \leq 5g(u-1)/2$.

Proof Let $g = 36n$, $n \equiv 1 \pmod 2$. Let $M_6 = \{6, 10, 14, 18, 22\}$. Then there exists a TD(6, u) for $u \geq 5$ with $u \notin M_6$ by Theorem 1.5 and, therefore, there exists a $\{6, u+1\}-$GDD of type $5^u u^1$ by Construction 1.13, which is our master design. There exist a 4-GDD of type $(36n)^5 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 72n$ and 4-GDDs of type $(36n)^u a_0^{\ 1}$, $a_0 \equiv 0 \pmod 3$, $0 \leq a_0 \leq 18n(u-1)$ by above premise. Every point in a group of size 5 in the master design is assigned weight $36n$. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain weights like a and the new point weights like a_0. The result is a $4-$GDD of type $(180n)^u m^1$ for 4-GDD for $m \equiv 0 \pmod 3$, $0 \leq m \leq 72n(u-1) + 18n(u-1) = 90n(u-1)$, $u \geq 5$ and $u \notin M_6$.

There exists a 4-RGDD of type 30^u, $u \in M_6$ by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $30^u(10(u-1))^1$ which is our master design. There exists a 4-GDD of type $(6n)^4 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 9n$ by Theorem 1.3 ($3 \leq a \leq 9$ for $n = 1$). In the last group of the master design the points obtain appropriate weights like a. All other points get weight $6n$. The result is a 4-GDD of type $(180n)^u m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 90n(u-1)$ ($30(u-1) \le m \le 90(u-1)$ for $n=1$).

For $n=1$ there exists a 4-RGDD of type 12^u, $u \in M_6$ by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $12^u(4(u-1))^1$ as our master design. There exists a 4-GDD of type $15^4 a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 21$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights like a. All other points obtain weight 15. The result is a 4-GDD of type $(180n)^u m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 84(u-1)$, $u \in M_6$.

The assertion follows for $u=4$ by Theorem 1.3. □

4. 4-GDDs of type $g^5 m^1$ and $g^6 m^1$

Since the case of $u=6$ is an exception in many recursive constructions in Section 3, all 4-GDDs of type $g^6 m^1$ with $g \equiv 0 \pmod 6$ are constructed in this section.

Lemma 4.1. *There exists a 4-GDD of type* $g^6 m^1$ *for* $g \equiv 0 \pmod{180}$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le g(6-1)/2$.

Proof There exists a 4-RGDD of type 30^6 by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $30^6(10(6-1))^1$ as our master design. There exists a 4-GDD of type $(6n)^4 a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 9n$ by Theorem 1.3 ($3 \le a \le 9$ for $n=1$). In the last group of the master design the points obtain appropriate weights like a. All other points receive weight $6n$. The result is a 4-GDD of type $(180n)^6 m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 90n(u-1)$ ($30(u-1) \le m \le 90(u-1)$ for $n=1$).

For $n=1$, there exists a 4-RGDD of type 12^6 by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $12^6(4(6-1))^1$, as our master design. There exists a 4-GDD of type $15^4 a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 21$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights like a. All other points obtain weight 15. The result is a 4-GDD of type $180^6 m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 84(u-1)$. □

Lemma 4.2. *There exists a 4-GDD of type* $18^6 m^1$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 9(6-1)$.

Proof There exists a 4-GDD of type $6^6 a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 15$ by Theorem 1.4 (6). Therefore, there exists a 4-GDD of type $18^6 a^1$, $a \equiv 0 \pmod 9$, $0 \le a \le 45$ by WFC. A 4-GDD of type $18^6 21^1$ is given in [33]. There exists a $\{3, 4\}$-LGDD$_{18}$ of type 1^6 with all blocks of size 3 partitioned into m 3-pc for $m \in \{3, 6, 12, 15\}$ by Lemma 2.3, which results in a $\{3, 4\}$-GDD of type 18^6 with all blocks of size 3 partitioned into m 3-pc for $m \in \{3, 6, 12, 15\}$ by Theorem 1.16. The assertion follows with Lemma 2.4. □

Lemma 4.3. *There exists a 4-GDD of type* $36^6 m^1$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 18(6-1)$.

Proof There exists a TD(7, 7) by Theorem 1.5 and we obtain a $\{7, 8\}$-GDD of type $6^7 7^1$ by Construction 1.13. Deleting all points from one group of size 6 we get a $\{6, 7, 8\}$-GDD of type $6^6 7^1$ as our master design. There exist 4-GDDs of types $6^5 a^1$, $6^6 a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 12$, $6^6 a_0^1$, $6^7 a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \le a_0 \le 15$ by Theorem 1.4 (6). We assign weight 6 to every point in a group of size 6 in the master design. The points in the group of size 7 obtain appropriate weights. The result is a 4-GDD of type $36^6 m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 12 \cdot 6 + 15 = 87$. There exists a 4-GDD of type $36^6 90^1$ by Theorem 1.2.□

Lemma 4.4. *There exists a 4-GDD of type* $90^6 m^1$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 45(6-1)$.

Proof There exists a 4-GDD of type $90^6 m^1$, $m \equiv 0 \pmod 3$ with $0 \le m \le 207$ by [26] (Lemma 6.2). A 4-GDD of type $90^6 216^1$ comes from a 4-GDD of type $30^6 72^1$ [26] and a 4-GDD of type $90^6 225^1$ exists by Theorem 1.2. The assertion follows with Lemma 2.5. □

Lemma 4.5. *Let* $M_6 = \{6, 10, 14, 18, 22\}$. *There exists a 4-GDD of type* $60^u m^1$ *for* $u \in M_6$ *and* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 24(u-1)+18$. *There exists a 4-GDD of type* $60^u m^1$ *for* $u \in M_6$ *and* $m \equiv 0 \pmod{15}$ *with* $0 \le m \le 30(u-1)$.

Proof There exists a TD(6, $u+1$), $u \in M_6$ by Theorem 1.5, and therefore there exists a $\{6, u+2\}$-GDD of type $5^{u+1}(u+1)^1$ by Construction 1.13. Removing a group of size 5, we obtain a $\{5, 6, u+1, u+2\}-$GDD of type $5^u(u+1)^1$. There exist 4-GDDs of types 12^4a^1, 12^5a^1, $a \equiv 0 \pmod 3$, $0 \le a \le 18$, and 4-GDDs of types $12^u a_0^1$, $12^{u+1}a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \le a_0 \le 6(u-1)$ by Theorem 1.4 (7). The points in the group of size $u+1$ obtain appropriate weights. The u "old" points obtain weights like a and the new point gets weights like a_0. The result is a 4-GDD of type $60^u m^1$, $u \in M_6$ and $m \equiv 0 \pmod 3$ with $0 \le m \le 18u + 6(u-1) = 24(u-1)+18$.

There exists a 4-GDD of type $12^u m^1$, $u \in M_6$ and $m \equiv 0 \pmod 3$ with $0 \le m \le 6(u-1)$ by Theorem 1.4, which is our master design. We assign each point weight 5, apply a 4-GDD of type 5^4 (Theorem 1.1) and obtain a 4-GDD of type $60^u(5m)^1$, which is the second assertion. □

Lemma 4.6. *There exists a 4-GDD of type* 60^6m^1 *for* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 150$.

Proof There exists a 4-GDD of type $60^6m_0^1$ for $m_0 \equiv 0 \pmod 3$, $0 \le m_0 \le 150$ by Lemma 4.5, Lemma 2.6 and Theorem 1.2. □

Theorem 4.7. *Let* $g \equiv 0 \pmod 6$. *Then there exists a 4-GDD of type* g^6m^1 *if, and only if,* $m \equiv 0 \pmod 3$ *with* $0 \le m \le g(6-1)/2$.

Proof There exists a 4-GDD of type $(6n)^6a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 15n$, $n \in \{1, 2\}$ by Theorem 1.5, which we apply as ingredient design. Let $M_7 = \{2, 3, 4, 5, 6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$. By Theorem 1.5 there exists a TD$(7, h)$ for $h \notin M_7$. This is a 7-GDD of type $h^7 = h^6h^1$ which we use as master design. In the last group of the master design the points obtain appropriate weights. All other points obtain weight $6n$. The result is a 4-GDD of type $(6nh)^6m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 3nh(6-1)$.

We obtain a 4-GDD of type $(6h)^6m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 3h(6-1)$ for $n=1$ and $h \notin M_7$. The remaining cases are shown in the following table.

nh	$6nh = g$	source of $(6nh)^6a^1$; $n \cdot h$
1	6	Theorem 1.4

2	12	Theorem 1.4
3	18	Lemma 4.2
4	24	Theorem 1.17
5	30	[26] (Theorem 4.6)
6	36	Lemma 4.3
10	60	Lemma 4.6
14	84	2·7
15	90	Lemma 4.4
18	108	2·9
20	120	Theorem 1.17
22	132	2·11
26	156	2·13
30	180	Lemma 4.1
34	204	2·17
38	228	2·19
46	276	2·23
60	360	Theorem 3.1. □

Lemma 4.8. *There exists a 4-GDD of type* $18^5 m^1$ *if, and only if,* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 18(5-1)/2 = 36$ *possibly excepting* $m \in \{3, 33\}$.

Proof There exists a 4-GDD of type $18^5 m^1$ for $m \in \{0, 6, 9, 18, 24, 27, 30, 36\}$ by [26] Lemmas 5.3 and 5.4. There exists a 4-GDD of type $18^5 m^1$ for $m \in \{12, 15, 21\}$ by Lemma 2.19. □

Theorem 4.9. *A 4-GDD of type* $36^u m^1$ *exists if, and only if,* $u \ge 4$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 18(u-1)$.

Proof There exists a 4-RGDD of type 12^u, $u \ge 4$ by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $12^u(4(u-1))^1$, as our master design. There exists a 4-GDD of type $3^4 a^1$, $a \in \{0, 3\}$ by Theorem 1.4 (3). In the last group of the master design the points obtain appropriate weights like a. All other points get weight 3. The result is a 4-GDD of type $36^u m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 12(u-1)$, $u \ge 4$.

There exists a TD(4, u) for $u \ge 4$, $u \ne 6$ by Theorem 1.5. Remove a point and use this point to redefine the groups. Complete the groups of size u with a new point. This gives a $\{4, u+1\}$ - GDD of type $3^u u^1$ as the master design. There exists a 4-GDD of type $12^u a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \le a_0 \le 6(u-1)$ by Theorem 1.4 (7). We give every point in a group of size 3 in the master design the weight

12. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain 12 as weight and the new point weights as a_0. The result is a 4-GDD of type $36^u m^1$, $u \geq 4$, $u \neq 6$, $m \equiv 0 \pmod 3$ and $12(u-1) \leq m \leq 12(u-1)+6(u-1)=18(u-1)$. The case of $u=6$ is solved in Theorem 4.7. □

Theorem 4.10. *Let* $g \equiv 0 \pmod 6$. *Then there exists a 4-GDD of type* $g^5 m^1$ *if, and only if,* $m \equiv 0 \pmod 3$ *with* $0 \leq m \leq g(5-1)/2$, *possibly excepting* $g=18$ *and* $m \in \{3, 33\}$.

Proof There exists a 4-GDD of type $(6n)^5 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 12n$, $n \in \{1, 2\}$ by Theorem 1.4, which we apply as ingredient design. Let $M_6 = \{6, 10, 14, 18, 22\}$. By Theorem 1.5 there exists a TD(6, h) for $h \geq 5$, $h \notin M_6$. This is a 6-GDD of type $h^6 = h^5 h^1$ which we use as master design. In the last group of the master design the points obtain appropriate weights. All other points obtain weight $6n$. The result is a 4-GDD of type $(6nh)^5 m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 3nh(5-1)$.
We receive a 4-GDD of type $(6h)^5 m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 3h(5-1)$ for $n=1$ and $h \geq 5$, $h \notin M_6$. The case of $g=18$ is shown in Lemma 4.8. The remaining cases are shown in the following table:

nh	$6nh=g$	source of $(6nh)^5 a^1$; $n \cdot h$
1	6	Theorem 1.4
2	12	Theorem 1.4
4	24	Theorem 1.17
6	36	Theorem 4.9
10	60	2·5
14	84	2·7
18	108	2·9
22	132	2·11. □

5. New 4-GDDs of type $g^u m^1$

Lemma 5.1. *Let* $g \equiv 12 \pmod{24}$. *There exists a 4-GDD of type* $g^{10} m^1$ *for* $m \equiv 0 \pmod 3$ *with* $0 \leq m \leq (g(10-1)/2)-18$.

Proof There exists a 4-RGDD of type 4^{10} by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $4^{10}12^1$, as our master design. Let $n=2i+1$, $i\geq 1$. There exists a 4-GDD of type $(3n)^4a^1$, $a\equiv 0 \pmod 3$, $0\leq a\leq 3(2i+1)3/2=9i+4.5$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights. All other points are assigned weight $3n$. The result is a 4-GDD of type $(12n)^{10}m^1$, $n\geq 3$, $m\equiv 0 \pmod 3$ with $0\leq m\leq 12(9i+3)=9(12i+4)=\{12(2i+1)-4\}\,9/2=12(2i+1)\,9/2-18$.
There exists a 4-GDD of type $12^{10}m^1$, $m\equiv 0 \pmod 3$ with $0\leq m\leq 6(10-1)$ by Theorem 1.4. □

Theorem 5.2. *Let* $g\equiv 12 \pmod{24}$. *There exists a 4-GDD of type* $g^u m^1$ *for* $u\geq 4$, $u\equiv 0,1,3 \pmod 4$, $m\equiv 0 \pmod 3$ *with* $0\leq m\leq g(u-1)/2-3\lfloor u/2\rfloor$.

Proof There exists a TD(5, u) for $u\geq 4$ and $u\notin\{6,10\}$ by Theorem 1.5. Remove a point and use this point to redefine the groups. Complete all groups of size u by adding a new point. This gives a $\{5,u+1\}$-GDD of type $4^u u^1$ as master design. Let $n=2i+1$, $i\geq 1$, therefore $n\geq 3$. There exist a 4-GDD of type $(3n)^4a^1$, $a\equiv 0 \pmod 3$, $0\leq a\leq 9n/2$ by Theorem 1.3, and a 4-GDD of type $(3n)^u a_0{}^1$, $a_0\in\{0, 3n(u-1)/2\}$ by Theorem 1.2 for u unequal 2 modulo 4. Every point in a group of size 4 in the master design is given the weight $3n$. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain weights like a and the new point gets weights like a_0. The result is a 4-GDD of type $(12n)^u m^1$ for $n\geq 3$, $u\geq 4$, $u\notin\{6,10\}$ and $m\equiv 0 \pmod 3$, $0\leq m\leq 9n(u-1)/2+3n(u-1)/2-3\lfloor u/2\rfloor=12n(u-1)/2-3\lfloor u/2\rfloor$. $\lfloor ...\rfloor$ is the integer part of the value. Each value of m can be combined, because the range of a has no gap.
There exists a 4-GDD of type $12^u m^1$ for $u\geq 4$, and $m\equiv 0 \pmod 3$, $0\leq m\leq 6(u-1)$ by Theorem 1.4 (7). □

Theorem 5.3. *Let* $g\equiv 0 \pmod{24}$. *There exists a 4-GDD of type* $g^u m^1$ *if, and only if,* $u\geq 4$, $m\equiv 0 \pmod 3$ *with* $0\leq m\leq g(u-1)/2$.

Proof There exists a TD(5, u) for $u\geq 4$ and $u\notin\{6,10\}$ by Theorem 1.5. Remove a point and use this point to redefine the groups. Complete all groups of size u by adding a new point. This gives a $\{5,u+1\}$ - GDD of type $4^u u^1$ as master design. There exist a 4-GDD of type $(6n)^4a^1$, $a\equiv 0 \pmod 3$,

$0 \le a \le 9n$, $n \ge 2$ by Theorem 1.3, and a 4-GDD of type $(6n)^u a_0^1$, $a_0 \in \{0, 3n(u-1)\}$ by Theorem 1.2. Every point in a group of size 4 in the master design is given weight $6n$. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain weights like a and the new point gets weights like a_0. The result is a 4-GDD of type $(24n)^u m^1$ for $n \ge 2$, $u \ge 4$, $u \notin \{6, 10\}$ and $m \equiv 0 \pmod 3$, $0 \le m \le 9n(u-1) + 3n(u-1)$ $= 12n(u-1)$. Each value of m can be combined, because the range of a has no gaps.

There exists a 4-GDD of type $(24n)^6 m^1$, $m \equiv 0 \pmod 3$, $0 \le m \le 12n(u-1)$ by Theorem 4.7.

There exists a 4-RGDD of type 4^{10} by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $4^{10}12^1$, as our master design. There exists a 4-GDD of type $(6n)^4 a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 9n$, $n \ge 2$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights. All other points are assigned weight $6n$. The result is a 4-GDD of type $(24n)^{10} m^1$, $n \ge 2$, $m \equiv 0 \pmod 3$ with $0 \le m \le 12 \cdot 9n = 12n(10-1)$.

There exists a 4-GDD of type $24^u m^1$ for $u \ge 4$, and $m \equiv 0 \pmod 3$, $0 \le m \le 12(u-1)$ by Theorem 1.17. □

Theorem 5.2 and Theorem 5.3 mean that we have the most values of m for $g \equiv 0 \pmod{12}$. There are better results in some special cases.

Theorem 5.4. *A 4-GDD of type* $108^u m^1$ *exists if, and only if,* $u \ge 4$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 54(u-1)$.

Proof All 4-GDDs of type $36^u m^1$ exist by Theorem 4.9. Therefore, the assertion follows with Theorem 3.2 and Theorem 4.7. □

Theorem 5.5. *A 4-GDD of type* $180^u m^1$ *exists if, and only if,* $u \ge 4$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 90(u-1)$.

Proof All 4-GDDs of type $36^u m^1$ exist by Theorem 4.9. Therefore, the assertion follows with Theorem 3.4. □

Theorem 5.6. *A 4-GDD of type* $192^u m^1$ *exists if, and only if,* $u \ge 4$, $m \equiv 0 \pmod 3$ *and* $0 \le m \le 96(u-1)$.

Proof All 4-GDDs of type $48^u m^1$ exist by Theorem 1.17. Therefore, the assertion follows with Theorem 3.3. □

Now we show the existence of 4-GDDs of type $(60n)^u m^1$. Let $M_6 = \{6, 10, 14, 18, 22\}$. Then there exists a TD(6, u) for $u \geq 5$ with $u \notin M_6$ by Theorem 1.5.

Theorem 5.7. *Let* $g = 60n$, $n \geq 1$. *Then there exists a 4-GDD of type* $g^u m^1$ *if, and only if,* $u \geq 4$, $m \equiv 0 \pmod 3$ *with* $0 \leq m \leq g(u-1)/2$, *possibly excepting*

$u = 14$,	$n = 1$ *and* $333 \leq m \leq 387$;
	$n > 1$ *odd and* $81 \leq m \leq 390n-3$;
$u = 18$,	$n > 1$ *odd and* $105 \leq m \leq 360n-3$.

Proof Let $n \geq 1$. A 4-GDD of type $(60n)^4 m^1$ exists for $m \equiv 0 \pmod 3$ with $0 \leq m \leq 90n$ exists by Theorem 1.3. Let $u \geq 5$, $u \notin M_6$ then there exists a TD(6, u), and, therefore, there exists a $\{6, u+1\}$-GDD of type $5^u u^1$ by Construction 1.13, which is our master design. There exist a 4-GDD of type $(12n)^5 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 24n$ by Theorem 4.10. There exists a 4-GDD of type $(12n)^u a_0{}^1$, $a_0 \in \{0, 6n(u-1)\}$ by Theorem 1.2 . We assign weight $12n$ to every point in each group of size 5 in the master design. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain weights like a and the new point gets weights like a_0. The result is a 4-GDD of type $60^u m^1$ for $u \geq 5$, $u \notin M_6$ and $m \equiv 0 \pmod 3$ with $0 \leq m \leq 24n(u-1) + 6n(u-1) = 30n(u-1)$. Each value of m can be obtained, since the range of a has no gap. The case $u = 6$ is handled in Theorem 4.7.
Let $u = 10$: There exists a 4-GDD of type $60^{10} m^1$ for $m \equiv 0 \pmod 3$ with $0 \leq m \leq 24(10-1) + 18 = 234$ by Lemma 4.5.
There exists a 4-RGDD of type 10^{10} by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $10^{10} 30^1$ as our master design. There exists a 4-GDD of type $6^4 a^1$, $a \in \{3, 6, 9\}$ by Theorem 1.4. In the last group of the master design the points obtain appropriate weights. All other points get weight 6. The result is a 4-GDD of type $60^{10} m^1$, $m \equiv 0 \pmod 3$ and $90 \leq m \leq 30 \cdot 9$.
There exists a 4-RGDD of type 10^{10} by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $10^{10} 30^1$ as our master design. Let $n \geq 2$ there exists a 4-GDD of type $(6n)^4 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 9n$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights. All

other points weight $6n$. The result is a 4-GDD of type $(60n)^{10}m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 30n(10-1)$.

$u=14$: There exists a 4-GDD of type $60^{14}m^1$ for $m \equiv 0 \pmod 3$ with $0 \le m \le 24(14-1)+18=330$ by Lemma 4.5.

For n even, there exists a 4-GDD of type $(60n)^{14}m^1$, $m \equiv 0 \pmod 3$ with $0 \le m \le 390n$ by Theorem 5.3.

For $n>1$ odd, there exists a 4-DGDD of type $(168, 12^{14})^{5n}$ by Theorem 1.7 and a 4-GDD of type $12^{14}m^1$, $m \equiv 0 \pmod 3$ with $0 \le m \le 78$ by Theorem 1.4 (7) and therefore, a 4-GDD of type $(60n)^{14}m^1$, $m \equiv 0 \pmod 3$ with $0 \le m \le 78$ by Construction 1.8.

$u=18$: There exists a 4-GDD of type $60^{18}m^1$ for $m \equiv 0 \pmod 3$ with $0 \le m \le 24(18-1)+18=426$ by Lemma 4.5. There exist a 4-GDD of type 360^4 by Theorem 1.1 (the master design). There exists a 4-GDD of type $60^6 m_0{}^1$ for $m_0 \equiv 0 \pmod 3$, $0 \le m_0 \le 150$ (the ingredient design) by Theorem 4.7. Adjoin m_0 infinite points to the last group of the master design and fill all other groups of the master design with the ingredient design. The result is a 4-GDD of type $60^{18}m^1$, $m \equiv 0 \pmod 3$, $360 \le m \le 510$.

For n even, there exists a 4-GDD of type $(60n)^{18}m^1$, $m \equiv 0 \pmod 3$ with $0 \le m \le 510n$ by Theorem 5.3.

There exists a 4-GDD of type $(60n)^6 m_0{}^1$ for $m_0 \equiv 0 \pmod 3$, $0 \le m_0 \le 150n$ (the ingredient design) by Theorem 4.7. There exist a 4-GDD of type $(360n)^4$ by Theorem 1.1 (the master design). Adjoin m_0 infinite points to the last group of the master design and fill all other groups of the master design with the ingredient design, whereas the infinite points form the group of size m_0. The result is a 4-GDD of type $(60n)^{18}m^1$, $m \equiv 0 \pmod 3$, $360n \le m \le 510n$.

For $n>1$ odd, there exists a 4-DGDD of type $(216, 12^{18})^{5n}$ by Theorem 1.7 and a 4-GDD of type $12^{18}m^1$, $m \equiv 0 \pmod 3$ with $0 \le m \le 102$ by Theorem 1.4 (7) and therefore, a 4-GDD of type $(60n)^{18}m^1$, $m \equiv 0 \pmod 3$ with $0 \le m \le 102$ by Construction 1.8.

$u=22$: There exists a 4-RGDD of type 2^{22} by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $2^{22}14^1$ as our master design. Let $n \ge 1$. There exists a 4-GDD of type $(30n)^4 a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 45n$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights. All other points weight $30n$. The result is a 4-GDD of type $(60n)^{22}m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 14 \cdot 45n = 30n(22-1)$. □

Next we show the existence of 4-GDDs of type $20^u m^1$.

Lemma 5.8. *There exists a 4-GDD of type $20^u m^1$ for each $u \ge 12$, $u \equiv 0 \pmod 3$, $m \equiv 2 \pmod 3$ and $2 \le m \le 10(u-1)$, possible except $u = 21$ and $m = 17$.*

Proof There exists a 4-GDD of type $60^{\hat{u}} m^1$, $\hat{u} \ge 4$, $\hat{u} \ne 14$, $m \equiv 0 \pmod 3$ and $0 \le m \le 30(\hat{u}-1)$ by Theorem 5.7. Adjoin 20 infinite points and fill all groups of size 60 with a 4-GDD of type 20^4 (Theorem 1.1), where the infinite points are filled in the group of size m. This results in a 4-GDD of type $20^{3\hat{u}}(m+20)^1$ for $\hat{u} \ge 4$, $\hat{u} \ne 14$, $m \equiv 0 \pmod 3$ and $20 \le m+20 \le 30(\hat{u}-1)+20 = 10(3\hat{u}-1)$.
There exists a 4-GDD of type $120^7 a^1$, $a \equiv 0 \pmod 3$ and $0 \le a \le 360$ by Theorem 1.17. Adjoin a_0 infinite points and fill all groups of size 120 with a 4-GDD of type $20^6 a_0^{\,1}$, $a_0 \in \{2, 50\}$, respectively (Theorem 1.2), in which the infinite points are filled in the group of size a. This gives a 4-GDD of type $20^{42} m^1$, $m \equiv 2 \pmod 3$ and $2 \le m = a + a_0 \le 360 + 50 = 410 = 10(42-1)$.
By Theorem 1.4 there exists a 4-GDD of type $5^u m^1$ for each $u \equiv 3 \pmod{12}$ and $m \equiv 5 \pmod 6$, $5 \le m \le 5(u-1)/2$; or $u \equiv 9 \pmod{12}$ and $m \equiv 2 \pmod 6$, $2 \le m \le 5(u-1)/2$; or $u \equiv 0 \pmod{12}$ and $m \equiv 2 \pmod 3$, $2 \le m \le (5(u-1)-3)/2$. Therefore, there exists a 4-GDD of type $20^u m^1$ for each $u \equiv 3 \pmod{12}$ and $m \equiv 5 \pmod 6$, $5 \le m \le 5(u-1)/2$; or $u \equiv 9 \pmod{12}$ and $m \equiv 2 \pmod 6$, $2 \le m \le 5(u-1)/2$; or $u \equiv 0 \pmod{12}$ and $m \equiv 2 \pmod 3$, $2 \le m \le (5(u-1)-3)/2$ by Corollary 1.9.
Particularly, there exists a 4-GDD of type $20^{12} m^1$, $m \in \{2, 5, 8, 11, 14, 17\}$.
Particularly, there exists a 4-GDD of type $20^{15} m^1$, $m \in \{5, 11, 17\}$. There exists a 4-GDD of type $2^{15} m^1$, $m \in \{2, 5, 8, 11, 14\}$ by Theorem 1.4. Therefore, there exists a 4-GDD of type $20^{15} m^1$ for $m \in \{2, 5, 8, 11, 14\}$ by Theorem 1.7 and Construction 1.8.
There exists a 4-GDD of type $2^{18} a^1$ for $a \equiv 2 \pmod 3$ and $2 \le a \le 18-1$ by Theorem 1.4. Therefore, there exists a 4-GDD of type $20^{18} a^1$ for $a \equiv 2 \pmod 3$ and $2 \le a \le 18-1$ by Theorem 1.7 and Construction 1.8.

Particularly, there exists a 4-GDD of type $20^{21}m^1$, $m \in \{2, 8, 14\}$. There exists a 4-GDD of type $2^{21}m^1$, $m \in \{2, 5, 8, 11, 14\}$ by Theorem 1.4. Therefore, there exists a 4-GDD of type $20^{21}m^1$ for $m \in \{2, 5, 8, 11, 14\}$ by Theorem 1.7 and Construction 1.8.
There exists a 4-GDD of type $2^u a^1$ for $u \geq 24$, $u \equiv 0 \pmod 3$ and $m \in \{2, 5, 8, 11, 14, 17\}$ by Theorem 1.4. Therefore, there exists a 4-GDD of type $20^u a^1$ for $u \geq 24$, $u \equiv 0 \pmod 3$ and $m \in \{2, 5, 8, 11, 14, 17\}$ by Theorem 1.7 and Construction 1.8. □

Lemma 5.9. *There exists a 4-GDD of type* $20^6 m^1$, $m \equiv 2 \pmod 3$ *with* $2 \leq m \leq 10(6-1)$.

Proof There exists a 4-GDD of type $4^6 a_0^{\ 1}$, $a_0 \equiv 1 \pmod 3$, $1 \leq a_0 \leq 10$ by Theorem 1.4. Therefore, there exists a 4-GDD of type $20^6 a^1$, $a \equiv 5 \pmod{15}$, $5 \leq a \leq 50$ by WFC. A 4-GDD of type $20^6 2^1$ exists by Theorem 2.1. All other needed 4-GDDs are given in Lemmas 2.16 and 2.17. □

Lemma 5.10. *There exists a 4-GDD of type* $20^9 m^1$ *if, and only if,* $m \equiv 2 \pmod 3$ *with* $2 \leq m \leq 80$, *possibly excepting* $m \in \{11, 17, 23\}$.

Proof There exists a 4-GDD of type $4^9 a_0^{\ 1}$, $a_0 \equiv 1 \pmod 3$, $1 \leq a_0 \leq 16$ by Theorem 1.4. Therefore, there exists a 4-GDD of type $20^9 a^1$, $a \equiv 5 \pmod{15}$, $5 \leq a \leq 80$ by WFC. There exists a 4-GDD of type $5^9 a_0^{\ 1}$, $a_0 \equiv 2 \pmod 6$, $2 \leq a_0 \leq 20$ by Theorem 1.4. Therefore, there exists a 4-GDD of type $20^9 a^1$, $a \in \{2, 8, 14, 20, 32, 56, 80\}$ by Construction 1.10. The assertion follows with Lemmas 2.18 and 2.19. □

The last three lemmas result in:

Theorem 5.11. *There exists a 4-GDD of type* $20^u m^1$ *if, and only if, either* $(u, m) = (3, 20)$ *or* $u \geq 6$ *and* $u \equiv 0 \pmod 3$, $m \equiv 2 \pmod 3$ *and* $2 \leq m \leq 10(u-1)$, *possibly excepting*
$u = 9$, $m \in \{11, 17, 23\}$ *and*
$u = 21$, $m = 17$.

Now we are going to deal with 4-GDDs of type $g^u m^1$ with $g \equiv 0 \pmod{36}$. Let $M_7 = \{5, 6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 39, 46, 60\}$.

Theorem 5.12. *Let* $g \equiv 36 \pmod{72}$. *There exists a 4-GDD of type* $g^u m^1$ *if, and only if,* $u \geq 4$, $m \equiv 0 \pmod{3}$ *with* $0 \leq m \leq g(u-1)/2$.

Proof A 4-GDD of type $36^u m^1$ exists if, and only if, $u \geq 4$, $m \equiv 0 \pmod{3}$ and $0 \leq m \leq 18(u-1)$ by Theorem 4.9. A 4-GDD of type $108^u m^1$ exists if, and only if, $u \geq 4$, $m \equiv 0 \pmod{3}$ and $0 \leq m \leq 54(u-1)$ by Theorem 5.4.
There exists a 4-RGDD of type 12^u, $u \geq 4$ by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $12^u(4(u-1))^1$ which is our master design. There exists a 4-GDD of type $(3n)^4 a^1$, $a \equiv 0 \pmod{3}$, $0 \leq a \leq (9n-3)/2$, $n \geq 5$, $n \equiv 1 \pmod{2}$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights like a. All other points obtain weight $3n$. The result is a 4-GDD of type $(36n)^u m^1$, $m \equiv 0 \pmod{3}$ and $0 \leq m \leq (18n-6)(u-1)$, $u \geq 4$, $n \geq 5$, $n \equiv 1 \pmod{2}$.
Let us deal separately with the two cases: u even and u odd.
Case 1- u even: There exists a 4-GDD of type $(36n)^u m^1$, $u \in \{4, 6\}$ for $m \equiv 0 \pmod{3}$ with $0 \leq m \leq 18n(u-1)$ by Theorem 1.3, and Theorem 4.7.
There exists a 4-RGDD of type $(6n)^u$, $n \geq 5$, $n \equiv 1 \pmod{2}$, $u \geq 4$, $u \equiv 0 \pmod{2}$ by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $(6n)^u(2n(u-1))^1$, as our master design. There exists a 4-GDD of type $6^4 a^1$, $a \in \{3, 6, 9\}$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights like a. All other points get weight 6. The result is a 4-GDD of type $(36n)^u m^1$, $m \equiv 0 \pmod{3}$ and $6n(u-1) \leq m \leq 18n(u-1)$, $n \geq 5$, $n \equiv 1 \pmod{2}$, $u \geq 4$, $u \equiv 0 \pmod{2}$.
Case 2 - u odd: There exists a 4-GDD of type $(36n)^5 m^1$ for $m \equiv 0 \pmod{3}$ with $0 \leq m \leq 18n(5-1)$ by Theorem 4.10.
Let $n \equiv 1 \pmod{2}$. There exists a TD(7, u) for $u \geq 7$ with $u \notin M_7$ by Theorem 1.5 and we obtain a $\{7, u+1\}$- GDD of type $6^u u^1$ by Construction 1.13 as our master design. There exists a 4-GDD of type $(6n)^6 a^1$, $a \equiv 0 \pmod{3}$, $0 \leq a \leq 3n(6-1)$ by Theorem 4.7. There exists a 4-GDD of type $(6n)^u \hat{a}^1$, $\hat{a} \in \{0, 3n(u-1)\}$ by Theorem 1.2. Every point in a group of size 6 in the master design is given weight $6n$. The points in the group of size u receive

appropriate weights. The result is a 4-GDD of type $(36n)^u m^1$, $m \equiv 0 \pmod 3$, $0 \le m \le 15n(u-1)+3n(u-1)=18n(u-1)$, $u \notin M_7$. Each value of m can be combined, since the range of a has no gap.
There exists a 4-GDD of type $(36n)^u m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le (18n-6)(u-1)$, $u \in \{15, 39\}$, $n \ge 5$, $n \equiv 1 \pmod 2$ by the first paragraph of the proof.
There exists a 4-GDD of type $(36\cdot 3n)^{\hat{u}} a^1$, $a \equiv 0 \pmod 3$ and $0 \le a \le 18\cdot 3n(\hat{u}-1)$, $\hat{u} \in \{5, 13\}$, $n \ge 5$, $n \equiv 1 \pmod 2$ see above. Adjoin $36n$ infinite points and fill all groups of size $36\cdot 3n$ with a 4-GDD of type $(36n)^4$ (Theorem 1.1), whereas the infinite points are filled in the group of size a. This gives a 4-GDD of type $(36n)^{3\hat{u}} m^1$ for $\hat{u} \in \{5, 13\}$, $n \ge 5$, $n \equiv 1 \pmod 2$, $m \equiv 0 \pmod 3$ and $36n \le a+36n \le 18\cdot 3n(\hat{u}-1)+36n = 18n(3\hat{u}-1)$. □

Theorem 5.13. *Let* $g \equiv 0 \pmod{36}$. *There exists a 4-GDD of type* $g^u m^1$ *if, and only if,* $u \ge 4$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le g(u-1)/2$.

Proof The assertion follows by Theorem 3.1, Theorem 5.12 and Theorem 1.2. □

Theorem 5.14. *Let* $g \equiv u \equiv 0 \pmod 6$, $u \ge 24$. *There exists a 4-GDD of type* $g^u m^1$ *if, and only if,* $m \equiv 0 \pmod 3$ *with* $0 \le m \le g(u-1)/2$.

Proof There exists a 4-GDD of type $(6g)^{\hat{u}} a^1$, $a \equiv 0 \pmod 3$ and $0 \le a \le 3g(\hat{u}-1)$ by Theorem 5.13 which is our master design. Adjoin a_0 infinite points and fill all groups of size $6g$ with a 4-GDD of type $g^6 a_0^{\,1}$, $a_0 \equiv 0 \pmod 3$ and $0 \le a_0 \le 5g/2$ (Theorem 4.7), whereas the infinite points are placed in the group of size a. This results in a 4-GDD of type $g^{6\hat{u}} m^1$ for $\hat{u} \ge 4$, $m \equiv 0 \pmod 3$ and $0 \le m = a + a_0 \le 3g(\hat{u}-1)+5g/2 = g(6\hat{u}-1)/2$. □

Now we are going to deal with 4-GDDs of type $84^u m^1$.
Let $M_8 = \{5, 6, 10, 12, 14, 15, 18, 20, 21, 22, 26, 28, 30,$
$33, 34, 35, 38, 39, 42, 44, 46, 51, 52, 54, 58, 60, 62, 66, 68, 74\}$.

Lemma 5.15. *There exists a 4-GDD of type* $84^u m^1$ *for* $m \equiv 0 \pmod 3$, $0 \le m \le 42(u-1)$, $u \ge 4$, $u \notin \{10, 14, 26, 38, 62, 74\}$.

Proof There exists a 4-GDD of type $84^u m^1$, $u \in \{4, 5, 6\}$ for $m \equiv 0 \pmod 3$ with $0 \le m \le 42(u-1)$ by Theorem 1.3, Theorem 4.7 and Theorem 4.10.
There exists a TD(8, u) for $u \ge 7$ with $u \notin M_8$ by Theorem 1.5 therefore, we obtain a $\{8, u+1\}$- GDD of type $7^u u^1$ by Construction 1.13 as our master design. There exists a 4-GDD of type $12^7 a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 36$, $12^u a_0^{\,1}$, $a_0 \equiv 0 \pmod 3$, $0 \le a_0 \le 6(u-1)$ by Theorem 1.4 (7). Every point in a group of size 7 in the master design is given weight 12. The points in the group of size u obtain appropriate weights. The result is a 4-GDD of type $84^u m^1$, $m \equiv 0 \pmod 3$, $0 \le m \le 36(u-1) + 6(u-1) = 42(u-1)$, $u \notin M_8$.
There exists a 4-RGDD of type 28^u, $u \ge 4$, $u \equiv 1 \pmod 3$ by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $28^u (28(u-1)/3)^1$, as our master design. There exists a 4-GDD of type $3^4 a^1$, $a \in \{0, 3\}$ by Theorem 1.1. In the last group of the master design the points obtain appropriate weights like a. All other points obtain weight 3. The result is a 4-GDD of type $84^u m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 28(u-1)$, $u \ge 4$, $u \equiv 1 \pmod 3$.
There exists a 4-RGDD of type 14^u, $u \ge 4$, $u \equiv 4 \pmod 6$, $u \notin \{10, 70, 82\}$ by Theorem 1.6. Completing the parallel classes results in a 5-GDD of type $14^u (14(u-1)/3)^1$ which is our master design. There exists a 4-GDD of type $6^4 a^1$, $a \in \{3, 6, 9\}$ by Theorem 1.3. In the last group of the master design the points obtain appropriate weights like a. All other points are given weight 6. The result is a 4-GDD of type $84^u m^1$, $m \equiv 0 \pmod 3$ and $14(u-1) \le m \le 42(u-1)$, $u \ge 4$, $u \equiv 4 \pmod 6$, $u \notin \{10, 70, 82\}$.
There exists a 4-GDD of type $84^u m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 42(u-1)$, $u > 12$, $u \equiv 0 \pmod 4$ by [16].
There exists a 4-GDD of type $252^{\hat{u}} m^1$ for $\hat{u} \ge 4$, $m \equiv 0 \pmod 3$ and $0 \le m \le 126(\hat{u}-1)$ by Theorem 5.12. Adjoin 84 infinite points and fill all groups of size 252 with a 4-GDD of type 84^4 (Theorem 1.1), whereas the infinite points are filled in the group of size m. This results in a 4-GDD of type $84^{3\hat{u}} (m+84)^1$, $m \equiv 0 \pmod 3$, $84 \le m + 84 \le 126(\hat{u}-1) + 84 = 42(3\hat{u}-1)$ for $\hat{u} \ge 4$ and therefore $0 \le m \le 42(u-1)$ for $u \in \{15, 21, 33, 39, 51\}$ by Theorem 5.2.
There exists a 4-DGDD of type $(216, 12^{18})^7$ by Theorem 1.7 and a 4-GDD of type $12^{18} m^1$, $m \equiv 0 \pmod 3$ with $0 \le m \le 102$ by Theorem 1.4 (7) and

therefore, a 4-GDD of type $84^{18}m^1$, $m \equiv 0 \pmod 3$ with $0 \le m \le 102$ by Construction 1.8 and therefore, $0 \le m \le 42(18-1)$ by the last paragraph.

There exists a 4-GDD of type $84^{30}m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 42(30-1)$ by Theorem 5.14.

There exists a 4-GDD of type $84^{35}m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 42(u-1)-51$ by Theorem 5.2.

There exists a 4-GDD of type $(7\cdot 84)^4(14\cdot 84)^1(7\cdot 84)^1 \equiv (7\cdot 84)^4(7\cdot 84)^1(14\cdot 84)^1$ by Theorem 1.12. There exists a 4-GDD of type 84^7a^1, $a \equiv 0 \pmod 3$, $0 \le a \le 42\cdot 6$ by this lemma. We add a points to the last group and fill all other groups with the above design. The result is a 4-GDD of type $84^{35}m^1$, $m \equiv 0 \pmod 3$, $1176 = 14\cdot 84 \le m \le 42\cdot 28 + 42\cdot 6 = 42(35-1)$. □

Theorem 5.16. *There exists a 4-GDD of type $84^u m^1$ for $u \ge 4$, $m \equiv 0 \pmod 3$, $0 \le m \le 42(u-1)$, except possibly when*

$u=10$ and $42(u-1)-18 < m < 42(u-1)$;

$u=14$ and $33(u-1)+9 < m < 42(u-1)$;

$u=26$ and $42(u-1)-42 < m < 42(u-1)$;

$u=38$, $m \in \{1161, 1164, 1167, 1170, 1173\}$, $42(u-1)-126 < m < 42(u-1)$;

$u=62$ and $42(u-1)-108 < m < 42(u-1)$;

$u=74$ and $42(u-1)-168 < m < 42(u-1)$.

Proof There exists a 4-GDD of type $84^u m^1$ for $m \equiv 0 \pmod 3$, $0 \le m \le 42(u-1)$, $u \ge 4$, $u \notin \{10, 14, 26, 35, 38, 62, 74\}$ by Lemma 5.15.

There exists a 4-GDD of type $84^{10}m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 42(10-1)-18$ by Lemma 5.1.

There exists a TD(8, $u+1$) for $u \in \{26, 35, 62, 74\}$ by Theorem 1.5 and we obtain a $\{8, u+2\}$-GDD of type $7^{u+1}(u+1)^1$ by Construction 1.13. Removing a group of size 7, we obtain a $\{7, 8, u+1, u+2\}$-GDD of type $7^u(u+1)^1$ as our master design. There exist 4-GDDs of types 12^6a^1, 12^7a^1, $a \equiv 0 \pmod 3$, $0 \le a \le 30$, $12^u a_0^{\,1}$, $12^{u+1}a_0^{\,1}$, $a_0 \equiv 0 \pmod 3$, $0 \le a_0 \le 6(u-1)$ by Theorem 1.4. Every point in a group of size 7 in the master design is given weight 12. The points in the group of size u obtain appropriate weights. The result is a 4-GDD of type $84^u m^1$, $u \in \{26, 35, 62, 74\}$, $m \equiv 0 \pmod 3$, $0 \le m \le 30u + 6(u-1) = 36(u-1)+30$.

There exists a TD(8, $u+2$) for $u\in\{14, 38\}$ by Theorem 1.5 and we obtain a $\{8, u+3\}$ - GDD of type $7^{u+2}(u+2)^1$ by Construction 1.13 as our master design. Removing two groups of size 7, we obtain a $\{6, 7, 8, u+1, u+2, u+3\}$ - GDD of type $7^u(u+2)^1$. There exist 4-GDDs of types 12^5a^1, 12^6a^1, 12^7a^1, $a\equiv 0 \pmod 3$, $0\le a\le 24$, $12^u a_0^{\,1}$, $12^{u+1}a_0^{\,1}$, $12^{u+2}a_0^{\,1}$, $a_0\equiv 0 \pmod 3$, $0\le a_0\le 6(u-1)$ by Theorem 1.4. Every point in a group of size 7 in the master design is given weight 12. The points in the group of size u obtain appropriate weights. The result is a 4-GDD of type $84^u m^1$, $u\in\{14, 38\}$, $m\equiv 0 \pmod 3$, $0\le m\le 24(u+1)+6(u-1)=30(u-1)+48$.

There exists a 4-GDD of type $(5\cdot 84)^4(10\cdot 84)^1(6\cdot 84)^1\equiv(5\cdot 84)^4(6\cdot 84)^1(10\cdot 84)^1$ by Theorem 1.12. There exist 4-GDDs of type 84^5a^1, 84^6a^1, $a\equiv 0 \pmod 3$, $0\le a\le 42\cdot 4$ by Lemma 5.15. We add a points to the last group and fill all other groups with the above designs. The result is a 4-GDD of type $84^{26}m^1$, $m\equiv 0 \pmod 3$, $840=33(26-1)+15\le m\le 840+42\cdot 4=42(26-1)-42$.

There exists a 4-GDD of type $(7\cdot 84)^4(14\cdot 84)^1(7\cdot 84)^1\equiv(7\cdot 84)^4(7\cdot 84)^1(14\cdot 84)^1$ by Theorem 1.12. There exists a 4-GDD of type 84^7a^1, $a\equiv 0 \pmod 3$, $0\le a\le 42\cdot 6$ by Lemma 5.15. We add a points to the last group and fill all other groups with the above design. The result is a 4-GDD of type $84^{35}m^1$, $m\equiv 0 \pmod 3$, $1176=34(35-1)+20\le m\le 42\cdot 28+42\cdot 6=42(35-1)$.

There exists a 4-GDD of type $(7\cdot 84)^4(14\cdot 84)^1(10\cdot 84)^1$ $\equiv(7\cdot 84)^4(10\cdot 84)^1(14\cdot 84)^1$ by Theorem 1.12. There exist 4-GDDs of types 84^7a^1, $84^{10}a^1$, $a\equiv 0 \pmod 3$, $0\le a\le 42\cdot 6$ by Lemma 5.15 and Lemma 5.1. We add a points to the last group and fill all other groups with the above designs. The result is a 4-GDD of type $84^{38}m^1$, $m\equiv 0 \pmod 3$, $1176=32(38-1)-8\le m\le 42\cdot 28+42\cdot 6=42(38-1)-3\cdot 42$.

There exists a 4-GDD of type $(12\cdot 84)^4(24\cdot 84)^1(14\cdot 84)^1$ $\equiv(12\cdot 84)^4(14\cdot 84)^1(24\cdot 84)^1$ by Theorem 1.12. There exist 4-GDDs of types $84^{12}a^1$, $84^{14}a^1$, $a\equiv 0 \pmod 3$, $0\le a\le 30\cdot 13+48=438$ by Lemma 5.15 and the proof above. We add a points to the last group and fill all other groups with the above designs. The result is a 4-GDD of type $84^{62}m^1$, $m\equiv 0 \pmod 3$, $2016\le m\le 42\cdot 48+438=42(62-1)-108$.

There exists a 4-GDD of type $(14\cdot 84)^4(28\cdot 84)^1(18\cdot 84)^1$ $\equiv(14\cdot 84)^4(18\cdot 84)^1(28\cdot 84)^1$ by Theorem 1.12. There exist 4-GDDs of types

$84^{14}a^1$, $84^{18}a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 30 \cdot 13 + 48 = 438$ by Lemma 5.15 and the proof above. We add a points to the last group and fill all other groups with the above designs. The result is a 4-GDD of type $84^{74}m^1$, $m \equiv 0 \pmod 3$, $42 \cdot 56 \le m \le 42 \cdot 56 + 438 = 42 \cdot 67 - 24 = 42(74-1) - 168$. □

Now we are going to deal with 4-GDDs of type $28^u m^1$.

Theorem 5.17. *There exists a 4-GDD of type $28^u m^1$ if, and only if, either $(u, m) = (3, 28)$ or $u \ge 6$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ and $1 \le m \le 14(u-1)$, except possibly when $u = 9$ and $m \in \{19, 25, 31\}$.*

Proof There exists a 7-GDD of type $7^7 \equiv 7^6 7^1$ by Theorem 1.5 as our master design. There exists a 4-GDD of type $4^6 a^1$, $a \equiv 1 \pmod 3$, $1 \le a \le 10$ by Theorem 1.4. In the last group of the master design the points obtain appropriate weights like a. All other points get weight 4. The result is a 4-GDD of type $28^6 m^1$, $m \equiv 1 \pmod 3$ and $7 \le m \le 10 \cdot 7 = 70 = 14(6-1)$.
There exists a 4-GDD of type $4^7 \equiv 4^6 4^1$ by Theorem 1.1 and there exists a 4-DGDD of type $(24, 4^6)^7$ by Theorem 1.7. Therefore, there exists a 4-GDD of type $28^6 4^1$ by Construction 1.8. There exists a 4-GDD of type $28^6 1^1$ by Theorem 1.2.
Recall the existence of a 4-DGDD of type $(36, 4^9)^7$ (Theorem 1.7). By Construction 1.8 and by WFC, we get the existence of 4-GDD of type $28^9 a^1$, $a \in \{1, 4, 7, 10, 13, 16, 28, 49, 70, 91, 112\}$.
There exists a 4-GDD of type $7^9 a_0^1$, $a_0 \equiv 4 \pmod 6$, $4 \le a_0 \le 28$ by Lemma 2.18. Therefore, there exists a 4-GDD of type $28^9 a^1$, $a \in \{4, 10, 16, 22, 28, 40, 64, 88, 112\}$ by Corollary 1.9. The assertion follows for $u = 9$ with the Lemmas 2.20 and 2.21.
There exists a 4-GDD of type $84^{\hat{u}} m^1$ for $\hat{u} \ge 4$, $\hat{u} \notin \{10, 14, 26, 38, 62, 74\}$, $m \equiv 0 \pmod 3$ and $0 \le m \le 42(\hat{u}-1)$ by Theorem 5.16. Adjoin 28 infinite points and fill all groups of size 84 with a 4-GDD of type 28^4 (Theorem 1.1), whereas the infinite points are filled in the group of size m. This results in a 4-GDD of type $28^{3\hat{u}}(m+28)^1$ for $\hat{u} \ge 4$, $\hat{u} \notin \{10, 14, 26, 38, 62, 74\}$, $m \equiv 0 \pmod 3$ and $28 \le m + 28 \le 42(\hat{u}-1) + 28 = 14(3\hat{u}-1)$.
There exists a 4-GDD of type $4^u m^1$ for each $u \ge 6$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ and $1 \le m \le 2(u-1)$ by Theorem 1.4. Therefore, there exists a

4-GDD of type $28^u m^1$ for each $u \geq 6$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ and $1 \leq m \leq 2(u-1)$ by Theorem 1.7 and Construction 1.8. Note that $2(u-1) \geq 28$ for $u \geq 15$.

There exists a 4-GDD of type $7^{12}25^1$ by [16]. Therefore, there exists a 4-GDD of type $28^{12}25^1$ by Corollary 1.9.

There exists a 4-GDD of type $168^{\hat{u}} a^1$ for $\hat{u} \geq 4$, $a \equiv 0 \pmod 3$ and $0 \leq a \leq 84(\hat{u}-1)$ by Theorem 5.3. In the proof above it was shown, that there exists a 4-GDD of type $28^6 a_0{}^1$, $a_0 \equiv 1 \pmod 3$ and $1 \leq a_0 \leq 70$. Adjoin a_0 infinite points and fill all groups of size 168 with the 4-GDD of type $28^6 a_0{}^1$, whereas the infinite points are filled in the group of size a. This gives a 4-GDD of type $28^{6\hat{u}} m^1$ for $\hat{u} \geq 4$, $m \equiv 1 \pmod 3$ and $1 \leq a + a_0{}^1 = m \leq 84(\hat{u}-1)+70$ $= 14(6\hat{u}-1)$. This solves the cases $u \in \{30, 42, 78, 114, 186, 222\}$. □

Now we are going to deal with 4-GDDs of type $10^u m^1$.

Lemma 5.18. *There exists a 4-GDD of type* $10^6 m^1$, $m \equiv 1 \pmod 3$ *with* $1 \leq m \leq 25$.

Proof There exists a 4-GDD of type $10^6 m^1$ for $m \in \{1, 25\}$ by Theorem 1.2, $m \in \{4, 7, 13, 16, 19\}$ by Lemma 2.1, $m = 22$ by Lemma 2.2 and $m = 10$ by Theorem 1.1. □

Lemma 5.19. *There exists a 4-GDD of type* $10^9 m^1$, $m \equiv 1 \pmod 3$ *with* $1 \leq m \leq 40$.

Proof There exists a 4-GDD of type $10^9 m^1$ for $m \in \{1, 40\}$ by Theorem 1.2, $m = 4$ by Lemma 2.14, $m \in \{7, 13, 19, 25, 31, 37\}$ by Lemma 2.8, $m = 10$ by Theorem 1.1 and $m \in \{16, 22, 28, 34\}$ by Lemma 2.7. □

Lemma 5.20. *There exists a 4-GDD of type* $10^u m^1$ *for* $m \in \{4, 7\}$, $u \geq 24$, $u \equiv 0 \pmod 6$ *and* $u \geq 33$, $u \equiv 3 \pmod 6$.

Proof There exists a 4-GDD of type $60^{\hat{u}}$, $\hat{u} \geq 4$ by Theorem 1.1 (the master design). Adjoin m infinite points to the master design and fill all groups of the master design with a 4-GDD of type $10^6 m^1$, $m \in \{4, 7\}$ from Lemma 2.1, whereas the infinite points form the group of size m. The result is a 4-GDD of type $10^{6\hat{u}} m^1$, $m \in \{4, 7\}$, $\hat{u} \geq 4$.

There exists a 4-GDD of type $60^{\hat{u}} 90^1$, $\hat{u} \geq 4$ by Theorem 5.7 (the master design). Adjoin m infinite points to the master design and fill all groups of size 60 of the master design with a 4-GDD of type $10^6 m^1$, $m \in \{4, 7\}$ from Lemma 2.1 and fill the group of size 90 with a 4-GDD of type $10^9 m^1$, $m \in \{4, 7\}$ from Lemma 5.25, whereas the infinite points form the group of size m. The result is a 4-GDD of type $10^{6\hat{u}+9} m^1$, $m \in \{4, 7\}$, $\hat{u} \geq 4$. □

Lemma 5.21. *There exists a 4-GDD of type* $10^u m^1$ *for each* $u \geq 12$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ *and* $10 \leq m \leq 5(u-1)$.

Proof There exists a 4-GDD of type $30^{\hat{u}} m^1$ for $\hat{u} \geq 4$, $\hat{u} \notin \{10, 14, 22\}$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 15(\hat{u}-1)$ by [26]. Adjoin 10 infinite points and fill all groups of size 30 with a 4-GDD of type 10^4 (Theorem 1.1), whereas the infinite points are filled in the group of size m. This gives a 4-GDD of type $10^{3\hat{u}}(m+10)^1$ for $\hat{u} \geq 4$, $m \equiv 0 \pmod 3$ and $10 \leq m+10 \leq 15(\hat{u}-1)+10 = 5(3\hat{u}-1)$.

There exists a 4-GDD of type $60^{\hat{u}} a^1$ for $\hat{u} \in \{5, 7, 11\}$, $a \equiv 0 \pmod 3$ and $0 \leq a \leq 30(\hat{u}-1)$ by Theorem 5.7. There exists a 4-GDD of type $10^6 a_0{}^1$, $a_0 \equiv 1 \pmod 3$ and $1 \leq a_0 \leq 25$ by Lemma 5.24. Adjoin a_0 infinite points and fill all groups of size 60 with the 4-GDD of type $10^6 a_0{}^1$, whereas the infinite points are filled in the group of size a. This gives a 4-GDD of type $10^{6\hat{u}} m^1$ for $\hat{u} \in \{5, 7, 11\}$, $m \equiv 1 \pmod 3$ and $1 \leq a + a_0 = m \leq 30(\hat{u}-1)+25 = 5(6\hat{u}-1)$. This solves the cases $u \in \{30, 42, 66\}$. □

Combining Theorem 1.1, Lemma 2.14 and the last four lemmas of this section, we obtain:

Theorem 5.22. *There exists a 4-GDD of type* $10^u m^1$ *if, and only if, either* $(u, m) = (3, 10)$ *or* $u \geq 6$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ *with* $1 \leq m \leq 5(u-1)$, *possibly excepting* $(u, m) \in \{(12, 4), (15, 7), (18, 4), (21, 7), (27, 7)\}$.

The main results of this paper are now summarized:

Theorem 5.23. *Let* $g \equiv 0, 24, 36, 48 \pmod{72}$. *There exists a 4-GDD of type* $g^u m^1$ *if, and only if,* $u \geq 4$, $m \equiv 0 \pmod 3$ *with* $0 \leq m \leq g(u-1)/2$.
Let $g = 60n$. *There exists a 4-GDD of type* $g^u m^1$ *if, and only if,* $u \geq 4$, $m \equiv 0 \pmod 3$ *with* $0 \leq m \leq g(u-1)/2$, *possibly excepting*

$u = 14$, $n = 1$ *and* $333 \leq m \leq 387$;
$n > 1$ *odd and* $81 \leq m \leq 390n - 3$;
$u = 18$, $n > 1$ *odd and* $105 \leq m \leq 360n - 3$.

Proof The cases $g \equiv 0, 24, 48 \pmod{72}$ follow by Theorem 5.3. The case $g \equiv 36 \pmod{72}$ is handled in Theorem 5.13. The case $g \equiv 0 \pmod{60}$ follows by Theorem 5.7. □

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