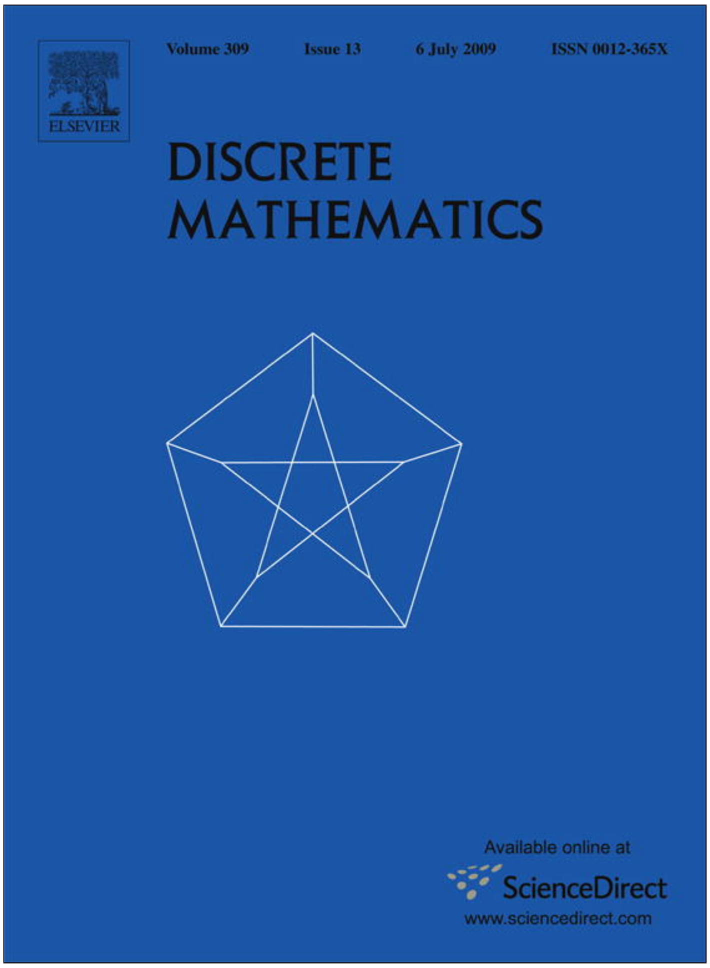
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Uniformly resolvable designs with index one, block sizes three and five and up to five parallel classes with blocks of size five

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Each parallel class of a uniformly resolvable design (URD) contains blocks of only one block size k (denoted k-pc). The number of k-pcs is denoted r_k. The necessary conditions for URDs with v points, index one, blocks of size 3 and 5, and $r_3, r_5 > 0$, are $v \equiv 15 \pmod{30}$. If $r_k > 1$, then $v \geq k^2$, and $r_3 = (v-1-4\cdot r_5)/2$. For $r_5 = 1$ these URDs are known as group divisible designs. We prove that these necessary conditions are sufficient for $r_5 = 3$ except possibly $v = 105$, and for $r_5 = 2, 4, 5$ with possible exceptions ($v = 105, 165, 285, 345$) New labeled frames and labeled URDs, which give new URDs as ingredient designs for recursive constructions, are the key in the proofs.

1. Introduction

Let v and λ be positive integers and let K and M be two sets of positive integers. A *group divisible design*, denoted $\mathrm{GDD}_\lambda(K, M; v)$, is a triple $(X, \boldsymbol{G}, \boldsymbol{B})$, where X is a set with v elements (called points), $\boldsymbol{G}$ is a set of subsets (called groups) of X, $\boldsymbol{G}$ partitions X, and $\boldsymbol{B}$ is a collection of subsets (called blocks) of X in such a way that:

1. $|B| \in K$ for each $B \in \boldsymbol{B}$;
2. $|G| \in M$ for each $G \in \boldsymbol{G}$;
3. $|B \cap G| \leq 1$ for each $B \in \boldsymbol{B}$ and $G \in \boldsymbol{G}$; and
4. each pair of elements of X from distinct groups is contained in exactly λ blocks.

The notation is similar to that used in [3]. Unless otherwise stated, the element set X of a design with v points is labeled $1, 2, \ldots, v$. If $\lambda = 1$, the index λ is omitted. If $K = \{k\}$, respectively $M = \{m\}$, then the $\mathrm{GDD}_\lambda(K, M; v)$ is simply denoted $\mathrm{GDD}_\lambda(k, M; v)$, respectively $\mathrm{GDD}_\lambda(K, m; v)$, which is also specified in "exponential" form as $K - GDD_\lambda$ of type $m^{v/m}$. A $\mathrm{GDD}_\lambda(K, 1; v)$ is called a *pairwise balanced design* and denoted $\mathrm{PBD}_\lambda(K; v)$.

Theorem 1.1 ([8,9]). *There exists a 4-GDD of type g^4m^1 with $m > 0$ if, and only if, $g \equiv m \equiv 0 \pmod 3$ and $0 < m \leq 3g/2$.*

In a $\mathrm{GDD}_\lambda(K, M; v)$ $(X, \boldsymbol{G}, \boldsymbol{B})$, a *parallel class* (pc) is a set of blocks which partitions X. If $\boldsymbol{B}$ can be partitioned into parallel classes, then the $\mathrm{GDD}_\lambda(K, M; v)$ is said to be *resolvable* and denoted $\mathrm{RGDD}_\lambda(K, M; v)$. Analogously, a resolvable $\mathrm{PBD}_\lambda(K; v)$ is denoted $\mathrm{RPBD}_\lambda(K; v)$. A parallel class is said to be *uniform* if it contains blocks of only one size k (k-pc). If all of the parallel classes of an $\mathrm{RPBD}_\lambda(K; v)$ ($\mathrm{RGDD}_\lambda(K, M; v)$) are uniform, the design is said to be *uniformly resolvable*. A uniformly resolvable design $\mathrm{RPBD}_\lambda(K; v)$ ($\mathrm{RGDD}_\lambda(K, M; v)$) is denoted by $\mathrm{URD}_\lambda(K; v)$ ($\mathrm{UGDD}_\lambda(K, M; v)$). If $\lambda = 1$, the index λ is omitted. In a $\mathrm{URD}_\lambda(K; v)$($\mathrm{UGDD}_\lambda(K, M; v)$) the number of uniformly resolution classes with blocks of size k, is denoted r_k.

All $\mathrm{RPBD}(5; v)$ are known apart from four possible exceptions.

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Theorem 1.2 *([1,2]). There exists a 5-RGDD of type* 5^{4t+1} *for* $t \geq 1,\ t \notin \{2, 17, 23, 32\}$.

Theorem 1.3 *([10,16]). There exists an RGDD*$(3, m; v)$ *if, and only if,* $v \equiv 0 \pmod{m},\ v \geq 3 \cdot m,\ v \equiv 0 \pmod{3}$ *and* $v - m \equiv 0 \pmod{2}$, *except when* $(v, m) = (6, 2), (12, 2), (18, 3)$.

A *resolvable transversal design* $\mathrm{RTD}_\lambda(k, g)$, is equivalent to an $\mathrm{RGDD}_\lambda(k, g; k \cdot g)$. That is, each block in an $\mathrm{RTD}_\lambda(k, g)$ contains a point from each group. A K-frame$_\lambda$ is a GDD $(X, \boldsymbol{G}, \boldsymbol{B})$ with index λ, in which $\boldsymbol{B}$ can be partitioned into holey parallel classes (each of which partitions $X \setminus G$ for some $G \in \boldsymbol{G}$). We use the usual exponential notation for the types of GDDs and frames. Thus a GDD or a frame of type $1^i 2^j, \ldots$ is one in which there are i groups of size 1, j groups of size 2, and so on. A K-frame is said to be *uniform* if each partial parallel class is of only one block size. It is said to be *completely uniform* if for each hole G the resolution classes which partition $X \setminus G$ are all of one block size. We use only $K = \{3, 5\}$. A $\{3, 5\}$-frame$_\lambda$ of type $(g;\ 3^{n_1} \cdot 5^{n_2})^u (m;\ 3^{n_3} \cdot 5^{n_4})^1$ has u groups of size g. Each group of size g has n_1 holey pcs of block size 3 and n_2 holey pcs of block size 5. The only group of size m has n_3 holey pcs of block size 3 and n_4 holey pcs of block size 5.

Theorem 1.4 *([12]). For* $k = 2$ *and* $k = 3$ *there exists a k-frame of type* h^u *if, and only if,* $u \geq k + 1,\ h \equiv 0 \pmod{k - 1}$, *and* $h \cdot (u - 1) \equiv 0 \pmod{k}$.

Later on, some incomplete group divisible designs will be used. An *incomplete group divisible design* (IGDD) with block sizes from a set K and index unity is a quadruple $(X, \boldsymbol{G}, H, \boldsymbol{B})$, which satisfies the following properties:

1. $\boldsymbol{G} = \{G_1,\ G_2, \ldots,\ G_n\}$ is a partition of the set X of points into subsets called *groups*;
2. H is a subset of X called the *hole*;
3. $\boldsymbol{B}$ is a collection of subsets of X with cardinality from K, called *blocks*, such that a group and a block contain at most one common point; and
4. every pair of points from distinct groups is either contained in H or occurs in a unique block but not both.

This design is denoted by IGDD$(K, M; v)$ of type T, where $M = \{|G_1|, |G_2|, \ldots, |G_n|\}$ and T is the multiset $\{(|G_i|, |G_i \cap H|) : 1 \leq i \leq n\}$. Sometimes, "exponential" notation is used to describe the type. An IGDD$(K, M; v)$ of type T is said to be *uniformly resolvable* and denoted by UIGDD$(K, M; v)$ of type T if its blocks can be partitioned into uniformly parallel classes and partial uniformly parallel classes, the latter partitioning $X \setminus H$. The numbers of uniformly parallel classes and partial uniformly parallel classes are denoted by r_k and r_k°, respectively. If $|G_i| = 1$ for $1 \leq i \leq n$ then the UIGDD is called an *incomplete uniformly resolvable design* IURD$(K; v)$ *with a hole* H.

Theorem 1.5 *([13]). For any* $v \equiv 3 \pmod{6}$ *and* $w \equiv 3 \pmod{6}$ *such that* $v \geq 3w$ *there is an RIPBD*$(3; v)$ *with a hole of size* w *and* $(w - 1)/2$ *holey 3-pcs, that is a 3-UIGDD of type* $1^{v-w} w^1$ *with* $r_3^\circ = (w - 1)/2$.

Some known results about URDs are summarized in the following. Rees introduced in [11] URDs and showed:

Theorem 1.6 *([11]). There exists a* URD $(\{2, 3\}; v)$ *with* $r_2, r_3 > 0$ *if, and only if,*

1. $v \equiv 0 \pmod{6}$;
2. $r_2 = v - 1 - 2 \cdot r_3 \left(r_3 = \frac{v-1-r_2}{2}\right)$; *and*
3. $1 \leq r_3 \leq \frac{v}{2} - 1$;

with the two exceptions $(v, r_3) \neq (6, 2), (12, 5)$.

Theorem 1.7 *([4,14]). There exists a design* URD $(\{3, 4\}; v)$ *with* $r_4 = 3$ *or* 5 *if, and only if,* $v \equiv 0 \pmod{12}$ *except when* $v = 12$.

Theorem 1.8 *(See [5] Theorems 1.1 and 3.6). The necessary conditions for the existence of a* URD$(\{3, 5\}; v)$ *with* $r_3, r_5 > 0$ *are:*

1. $v \equiv 15 \pmod{30}$;
2. *if* $r_k > 1$, *then* $v \geq k^2$; *and*
3. $r_5 = \frac{v-1-2\cdot r_3}{4}, \quad \left(r_3 = \frac{v-1-4\cdot r_5}{2}\right)$.

The third condition means that if r_3 is given, then r_5 is determined, and vice versa. The only known result with $r_5 > 0$ is a special case of Theorem 1.3. We take the groups as an additional parallel class to get the URD.

Theorem 1.9. *There exists a 3 - RGDD of type* $5^{3\cdot(2n+1)},\ n \geq 0$; *and also a* URD $(\{3, 5\}; v)$ *with* $r_5 = 1$ *for all* $v \equiv 15 \pmod{30}$.

In the next section labeled resolvable designs are introduced. We construct some new labeled uniformly resolvable designs which will be used as ingredients for our main recursive constructions in Section 3. The results will be given in the last two sections.

2. Labeled resolvable designs and direct constructions

The concept of labeled resolvable designs is needed in order to get direct constructions for resolvable designs. This concept was introduced by Shen [15,17,16].

Let $(X, \boldsymbol{B})$ be a (U)GDD$_\lambda(K, M; v)$ where $X = \{a_1, a_2, \ldots, a_v\}$ is totally ordered with ordering $a_1 < a_2 < \cdots < a_v$. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, it is supposed that $x_1 < x_2 < \cdots < x_k$. Let Z_λ be the group of residues modulo λ.

Let $\varphi : \boldsymbol{B} \to Z_\lambda^{\binom{k}{2}}$ be a mapping where for each $B = \{x_1, x_2, \ldots, x_k\} \in \boldsymbol{B}$, $k \in K$, $\varphi(B) = (\varphi(x_1, x_2), \ldots, \varphi(x_1, x_k), \varphi(x_2, x_3), \ldots, \varphi(x_2, x_k), \varphi(x_3, x_4), \ldots, \varphi(x_{k-1}, x_k))$, $\varphi(x_i, x_j) \in Z_\lambda$ for $1 \le i < j \le k$.

A (U)GDD$_\lambda(K, M; v)$ is said to be a *labeled (uniform resolvable) group divisible design*, denoted L(U)GDD$_\lambda(K, M; v)$, if there exists a mapping φ such that:

1. For each pair $\{x, y\} \subset X$ with $x < y$, contained in the blocks $B_1, B_2, \ldots, B_\lambda$, then $\varphi_i(x, y) \equiv \varphi_j(x, y)$ if, and only if, $i = j$ where the subscripts i and j denote the blocks to which the pair belongs, for $1 \le i, j \le \lambda$; and
2. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, $\varphi(x_r, x_s) + \varphi(x_s, x_t) \equiv \varphi(x_r, x_t) \pmod{\lambda}$, for $1 \le r < s < t \le k$.

Its blocks will be denoted in the following form:

$$(x_1 x_2 \cdots x_k;\ \varphi(x_1, x_2) \cdots \varphi(x_1, x_k)\varphi(x_2, x_3) \cdots \varphi(x_2, x_k)\varphi(x_3, x_4) \cdots \varphi(x_{k-1}, x_k)), \quad k \in K.$$

The above definition was first used in [14] and is a little bit more general than the definition by Shen [16] with $K = \{k\}$ or Shen and Wang [17] for transversal designs. As a special case of type 1^v, a labeled URD$_\lambda(K; v)$, is denoted by LURD$_\lambda(K; v)$. A labeled K-frame of type T and index λ is denoted K-LF$_\lambda$ of type T.

The following designs were constructed by a computer.

Example 2.1. The following is an example of an LURD$_3$ ({3, 5} ; 15) with $r_5 = 2$, where each row forms a uniformly parallel class:

(6 7 13; 0 1 1), (1 3 11; 1 0 2), (12 14 15; 2 2 0), (2 5 8; 2 2 0), (4 9 10; 2 1 2),
(6 7 14; 2 1 2), (5 9 11; 0 2 2), (8 13 15; 1 0 2), (3 4 12; 2 1 2), (1 2 10; 2 0 1),
(7 10 14; 2 1 2), (2 4 11; 2 2 0), (3 5 12; 0 0 0), (9 13 15; 2 2 0), (1 6 8; 0 2 2),
(1 4 13; 2 1 2), (6 8 10; 0 2 2), (5 14 15; 0 2 2), (2 9 12; 2 0 1), (3 7 11; 0 1 1),
(1 4 5; 1 2 1), (2 3 13; 1 1 0), (10 11 14; 2 1 2), (6 7 9; 1 0 2), (8 12 15; 1 1 0),
(1 9 10; 0 1 1), (4 6 14; 0 0 0), (3 5 8; 1 0 2), (2 7 15; 2 1 2), (11 12 13; 2 2 0),
(10 13 15; 0 1 1), (6 9 11; 1 2 1), (3 5 7; 2 2 0), (1 2 12; 1 2 1), (4 8 14; 1 2 1),
(2 4 8; 0 0 0), (7 9 15; 1 1 0), (3 6 11; 2 0 1), (5 10 12; 2 2 0), (1 13 14; 0 1 1),
(7 8 13; 2 2 0), (2 5 9; 1 0 2), (1 11 14; 1 2 1), (6 12 15; 2 0 1), (3 4 10; 1 1 0),
(2 3 6; 2 2 0), (4 8 9; 2 1 2), (1 11 15; 2 1 2), (7 12 14; 2 0 1), (5 10 13; 1 2 1),
(1 7 10; 2 2 0), (6 9 13; 2 0 1), (3 8 14; 1 0 2), (4 5 12; 0 1 1), (2 11 15; 0 0 0),
(7 8 11; 0 0 0), (3 10 12; 0 2 2), (1 5 6; 0 2 2), (4 9 15; 0 1 1), (2 13 14; 0 2 2),
(2 6 8; 0 1 1), (1 3 9; 0 2 2), (4 14 15; 1 2 1), (5 7 10; 2 0 1), (11 12 13; 1 0 2),
(4 6 11; 1 1 0), (3 13 14; 2 2 0), (8 9 10; 0 0 0), (1 5 15; 1 2 1), (2 7 12; 1 2 1),
(1 4 7; 0 1 1), (2 10 14; 0 0 0), (3 9 13; 1 1 0), (8 11 12; 2 2 0), (5 6 15; 1 0 2),
(5 8 13; 1 0 2), (10 11 15; 1 2 1), (3 9 14; 0 1 1), (1 6 12; 1 1 0), (2 4 7; 1 0 2),
(8 10 11; 1 1 0), (3 4 15; 0 0 0), (5 6 14; 0 2 2), (7 9 12; 0 0 0), (1 2 13; 0 2 2),
(1 3 7 8 15; 2 0 1 0 1 2 1 1 0 2), (4 6 10 12 13; 2 2 0 1 0 1 2 1 2 1), (2 5 9 11 14; 0 1 1 1 1 1 1 0 0 0),
(4 5 7 11 13; 2 0 2 0 1 0 1 2 0 1), (1 8 9 12 14; 0 1 0 0 1 0 0 2 2 0), (2 3 6 10 15; 0 1 2 2 1 2 2 1 1 0).

Example 2.2. An LURD$_3$ ({3, 5} ; 15) with $r_5 = 3$, where each row forms a uniformly parallel class:

(3 4 15; 1 2 1), (5 6 13; 1 0 2), (8 9 12; 2 2 0), (1 7 10; 2 2 0), (2 11 14; 0 2 2),
(6 7 12; 1 0 2), (10 11 15; 0 0 0), (2 4 8; 0 2 2), (3 13 14; 0 1 1), (1 5 9; 1 2 1),
(4 8 11; 0 1 1), (1 7 12; 0 0 0), (6 9 14; 0 1 1), (2 3 13; 1 2 1), (5 10 15; 2 0 1),
(2 5 10; 1 2 1), (7 9 11; 0 2 2), (1 3 4; 2 1 2), (6 12 15; 2 2 0), (8 13 14; 1 0 2),
(2 10 12; 0 0 0), (3 9 14; 2 2 0), (6 8 15; 0 0 0), (1 7 13; 1 0 2), (4 5 11; 0 0 0),
(11 12 13; 0 2 2), (4 5 7; 2 2 0), (6 8 14; 1 0 2), (1 2 15; 2 1 2), (3 9 10; 1 2 1),
(1 8 11; 2 1 2), (3 7 15; 2 1 2), (2 9 14; 2 1 2), (5 6 13; 2 2 0), (4 10 12; 2 1 2),
(1 3 9; 1 1 0), (4 6 12; 1 2 1), (10 13 15; 0 2 2), (7 8 14; 2 0 1), (2 5 11; 0 2 2),
(12 14 15; 2 1 2), (1 9 10; 0 0 0), (4 5 13; 1 2 1), (2 6 7; 0 0 0), (3 8 11; 0 0 0),
(7 10 14; 1 2 1), (2 3 6; 0 1 1), (4 9 13; 0 1 1), (11 12 15; 2 1 2), (1 5 8; 0 0 0),
(3 11 15; 1 0 2), (5 7 9; 1 0 2), (4 8 10; 1 0 2), (1 6 14; 1 0 2), (2 12 13; 2 0 1),
(5 11 14; 1 1 0), (3 4 7; 0 0 0), (1 6 15; 2 0 1), (8 10 13; 1 0 2), (2 9 12; 0 1 1),
(1 2 10; 0 1 1), (5 14 15; 0 1 1), (7 11 13; 0 1 1), (4 6 9; 2 1 2), (3 8 12; 2 0 1),
(6 8 10; 2 2 0), (1 11 14; 0 1 1), (5 7 12; 2 0 1), (2 3 13; 2 1 2), (4 9 15; 2 0 1),
(9 11 12; 1 2 1), (3 5 6; 0 0 0), (1 13 15; 1 2 1), (2 7 8; 1 1 0), (4 10 14; 1 1 0),
(7 8 9 13 15; 1 1 0 0 0 2 2 2 2 0), (1 2 4 6 11; 1 0 0 2 2 2 1 0 2 2), (3 5 10 12 14; 1 1 2 0 0 1 2 1 2 1),
(1 4 12 13 14; 2 2 2 2 0 0 0 0 0 0), (3 6 7 10 11; 2 1 0 2 2 1 0 2 1 2), (2 5 8 9 15; 2 0 1 1 1 2 2 1 1 0),
(1 3 5 8 12; 0 2 1 1 2 1 1 2 2 0), (6 9 10 11 13; 1 0 1 1 2 0 0 1 1 0), (2 4 7 14 15; 1 2 0 0 1 2 2 1 1 0).

Example 2.3. An LURD_3 ($\{3, 5\}$; 15) with $r_5 = 4$, where each row forms a uniformly parallel class:
(1 13 14; 0 1 1), (2 9 15; 0 0 0), (5 8 11; 1 2 1), (3 10 12; 2 0 1), (4 6 7; 1 2 1),
(3 7 14; 2 0 1), (1 4 11; 1 0 2), (6 8 15; 2 1 2), (5 10 13; 0 1 1), (2 9 12; 1 1 0),
(4 6 9; 0 2 2), (8 13 14; 1 1 0), (1 7 10; 1 0 2), (2 3 5; 2 2 0), (11 12 15; 1 0 2),
(2 4 13; 2 1 2), (5 7 14; 1 1 0), (6 10 15; 2 2 0), (3 9 11; 0 1 1), (1 8 12; 0 0 0),
(2 6 14; 2 0 1), (8 10 13; 0 2 2), (1 9 15; 2 1 2), (3 11 12; 2 2 0), (4 5 7; 1 0 2),
(7 12 15; 1 1 0), (1 4 5; 0 2 2), (2 8 10; 2 1 2), (9 11 14; 0 0 0), (3 6 13; 1 1 0),
(5 8 12; 2 0 1), (4 9 11; 1 0 2), (3 7 15; 1 1 0), (1 6 14; 1 0 2), (2 10 13; 0 0 0),
(9 10 15; 2 1 2), (1 5 11; 0 1 1), (2 3 4; 0 0 0), (7 12 13; 0 0 0), (6 8 14; 1 0 2),
(4 8 9; 2 0 1), (5 13 15; 2 0 1), (1 7 12; 0 2 2), (6 10 11; 0 1 1), (2 3 14; 1 2 1),
(3 5 10; 2 1 2), (4 6 7; 2 1 2), (9 12 13; 1 2 1), (2 8 11; 1 1 0), (1 14 15; 2 2 0),
(11 12 14; 2 2 0), (1 3 13; 2 1 2), (4 8 10; 0 1 1), (2 7 9; 0 2 2), (5 6 15; 2 2 0),
(10 11 14; 2 0 1), (2 5 15; 0 1 1), (3 6 12; 0 1 1), (7 8 9; 1 0 2), (1 4 13; 2 2 0),
(4 10 12; 2 1 2), (1 2 15; 1 0 2), (5 6 11; 0 0 0), (9 13 14; 0 2 2), (3 7 8; 0 0 0),
(1 2 7 10 11; 0 2 2 2 2 2 2 0 0 0), (5 6 9 12 13; 1 2 1 0 1 0 2 2 1 2), (3 4 8 14 15; 1 2 2 0 1 1 2 0 1 1),
(1 2 6 8 12; 2 2 2 1 0 0 2 0 2 2), (3 4 11 13 15; 2 0 0 2 1 1 0 0 2 2), (5 7 9 10 14; 0 1 1 2 1 1 2 0 1 1),
(1 3 5 8 9; 0 1 1 1 1 1 1 0 0 0), (2 6 7 11 13; 1 1 0 2 0 2 1 2 1 2), (4 10 12 14 15; 0 0 2 1 0 2 1 2 1 2),
(1 3 6 9 10; 1 0 0 1 2 2 0 0 1 1), (7 8 11 13 15; 2 1 2 2 2 0 0 1 1 0), (2 4 5 12 14; 1 1 0 1 0 2 0 2 0 1).

Example 2.4. An LURD_3 ($\{3, 5\}$; 15) with $r_5 = 5$, where each row forms a uniformly parallel class:
(1 2 6; 0 0 0), (3 13 14; 1 0 2), (4 5 9; 2 2 0), (7 8 12; 1 2 1), (10 11 15; 0 0 0),
(1 2 9; 1 2 1), (3 4 11; 0 0 0), (5 6 13; 2 2 0), (7 8 15; 0 1 1), (10 12 14; 0 1 1),
(1 3 5; 2 2 0), (2 4 6; 0 1 1), (7 14 15; 2 0 1), (8 10 12; 1 0 2), (9 11 13; 2 2 0),
(1 3 14; 0 2 2), (2 4 15; 2 1 2), (5 12 13; 2 1 2), (6 8 10; 1 0 2), (7 9 11; 2 0 1),
(1 5 15; 0 2 2), (2 12 13; 1 2 1), (3 4 8; 1 0 2), (6 7 11; 1 2 1), (9 10 14; 0 0 0),
(1 6 11; 2 0 1), (2 7 12; 1 2 1), (3 8 13; 1 0 2), (4 9 14; 0 1 1), (5 10 15; 0 1 1),
(1 6 11; 1 1 0), (2 7 12; 0 0 0), (3 8 13; 2 2 0), (4 9 14; 1 0 2), (5 10 15; 1 0 2),
(1 8 9; 2 0 1), (2 13 15; 1 0 2), (3 10 11; 1 2 1), (4 5 12; 1 2 1), (6 7 14; 2 0 1),
(1 8 15; 1 0 2), (2 3 10; 1 1 0), (4 11 12; 1 1 0), (5 7 9; 0 1 1), (6 13 14; 1 2 1),
(1 11 12; 2 0 1), (2 3 7; 2 2 0), (4 14 15; 2 1 2), (5 6 10; 1 2 1), (8 9 13; 0 1 1),
(1 12 14; 1 1 0), (2 9 10; 0 2 2), (3 5 7; 2 1 2), (4 6 8; 0 0 0), (11 13 15; 1 1 0),
(1 2 4 10 13; 2 0 2 2 1 0 0 2 2 0), (3 6 7 9 15; 2 2 2 1 0 0 2 0 2 2), (5 8 11 12 14; 1 1 0 2 0 2 1 2 1 2),
(1 3 9 12 15; 1 1 2 1 0 1 0 1 0 2), (2 5 6 8 14; 2 2 1 0 0 2 1 2 1 2), (4 7 10 11 13; 0 0 2 1 0 2 1 2 1 2),
(1 4 5 7 13; 1 1 2 1 0 1 0 1 0 2), (2 8 11 14 15; 2 0 2 2 1 0 0 2 2 0), (3 6 9 10 12; 0 1 2 0 1 2 0 1 2 1),
(1 4 7 8 10; 2 1 0 0 2 1 1 2 2 0), (2 3 5 11 14; 0 1 1 1 1 1 1 0 0 0), (6 9 12 13 15; 2 2 2 0 0 0 1 0 1 1),
(1 7 10 13 14; 0 1 0 0 1 0 0 2 2 0), (2 5 8 9 11; 0 0 2 2 0 2 2 2 2 0), (3 4 6 12 15; 2 1 2 2 2 0 0 1 1 0).

The main application of the labeled designs is to blow up the point set of a given design using the following theorem, which extends the work of [15] such that it is applicable to labeled (uniform resolvable) pairwise balanced designs.

Theorem 2.5 ([15,14]). *If there exists an* $L(U)GDD_\lambda(K, M; v)$ *(with* r_k^L *classes of size* k*, for each* $k \in K$*), then there exists an* $(U)GDD(K, \lambda \cdot M; \lambda \cdot v)$*, where* $\lambda \cdot M = \{\lambda \cdot g_i | g_i \in M\}$ *(with* $r_k = r_k^L$ *classes of size* k*, for each* $k \in K$*). If there exists a uniform frame* K*-*LF_λ *of type* T*, then there exists a uniform* K*-frame of type* $\lambda \cdot T$*, where* $\lambda \cdot T = \{\lambda \cdot g_i | g_i \in T\}$*.*

Proof. Let $(X, \boldsymbol{G}, \boldsymbol{B})$ be an $\text{LRGDD}_\lambda(K, M; v)$ where $X = \{a_1, a_2, \ldots, a_v\}$. Expanding each point $a_i \in X$ λ times gives the points $\left\{a_{i,0}, \ldots, a_{i,\lambda-1}\right\}$, $i = 1, \ldots, v$, in the new design. Any group with g_i points becomes a new group with $\lambda \cdot g_i$ points. Each labeled block $(x_1 x_2 \ldots x_k; \varphi(x_1, x_2) \ldots \varphi(x_1, x_k)\varphi(x_2, x_3) \ldots \varphi(x_2, x_k)\varphi(x_3, x_4) \ldots \varphi(x_{k-1}, x_k))$, $k \in K$, gives λ new blocks $\left\{x_{1,j}, x_{2,j+\varphi(x_1,x_2)}, \ldots, x_{k,j+\varphi(x_1,x_k)}\right\}$, $k \in K, j = 0, \ldots, (\lambda - 1)$ with indices calculated mod (λ) and all blocks taken together consist of different points. Therefore, each (holey) uniformly parallel class of the labeled design with blocks of size k gives a (holey) parallel class of the expanded design with blocks of the same size k. Since a frame consists of holey parallel classes, it can be expanded, too. For each pair $\{x, y\} \subset X$ with $x < y$ from different groups, let $B_1, B_2, \ldots, B_\lambda$ be the λ blocks containing $\{x, y\}$ and let $\varphi_i(x, y)$ be the values of $\varphi(x, y)$ corresponding to B_i, $1 \le i \le \lambda$. Due to the first condition all pairs $\left\{x_j, y_{j+\varphi_i(x,y)}\right\}$, $i = 1, \ldots, \lambda, j = 0, \ldots, (\lambda - 1)$, with indices calculated mod (λ), are different. □

A special case for URDs is shown in the following.

Corollary 2.6. *If there exists an* $\text{LURD}_\lambda(K; v)$ *with* r_k^L *classes of size* k*, for each* $k \in K$*, then there exists a* $\text{URD}(K \cup \{\lambda\} ; \lambda \cdot v)$ *with* $r_k = r_k^L$ *when* $k \neq \lambda$*, and* $r_\lambda = r_\lambda^L + 1$*, where we take* $r_\lambda^L = 0$ *if* $\lambda \notin K$*.*

Lemma 2.7. *A* URD ($\{3, 5\}$; 45) *with* $r_5 = 2, 3, 4, 5$ *exists.*

Proof. The designs $\text{LURD}_3(\{3, 5\} ; 15)$ with $r_5 = 2, 3, 4, 5$ are given in Examples 2.1–2.4. Therefore, the assertion follows by Corollary 2.6. □

Example 2.8. A uniform labeled frame $\{3, 5\}$-LF$_6$ of type $(1; 3^3)^5(2; 5^3)^1$, $\boldsymbol{G} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6, 7\}\}$; each row forms a holey uniformly parallel class:

(4 5 6; 1 5 4), (2 3 7; 1 4 3),
(3 4 7; 1 0 5), (2 5 6; 0 1 1),
(3 5 6; 5 1 2), (2 4 7; 0 1 1),
(1 5 7; 3 2 5), (3 4 6; 2 2 0),
(3 5 6; 1 0 5), (1 4 7; 5 5 0),
(1 3 6; 4 3 5), (4 5 7; 5 2 3),
(2 5 7; 5 3 4), (1 4 6; 0 1 1),
(2 4 7; 2 0 4), (1 5 6; 5 5 0),
(1 2 6; 0 2 2), (4 5 7; 2 3 1),
(3 5 7; 2 4 2), (1 2 6; 2 0 4),
(2 3 6; 5 3 4), (1 5 7; 4 4 0),
(2 5 6; 3 0 3), (1 3 7; 2 3 1),
(2 4 6; 1 5 4), (1 3 7; 1 0 5),
(3 4 6; 0 3 3), (1 2 7; 5 1 2),
(1 4 6; 2 4 2), (2 3 7; 3 5 2),
(1 2 3 4 5; 4 0 3 0 2 5 2 3 0 3),
(1 2 3 4 5; 3 3 1 1 0 4 4 4 4 0),
(1 2 3 4 5; 1 5 4 2 4 3 1 5 3 4).

Lemma 2.9. *There exists a $\{3, 5\}$-frame of type $(6; 3^3)^5(12; 5^3)^1$.*

Proof. A uniform $\{3, 5\}$-LF$_6$ of type $(1; 3^3)^5(2; 5^3)^1$ is given in Example 2.8. Therefore, the assertion follows by Theorem 2.5. □

3. Recursive constructions and first results

We now describe some constructions which we will use later. Filling groups and holes with PBDs or GDDs is known as basic construction [6,7]. Here, groups and holes are filled with URDs to get new URDs.

Construction 3.1 (*Breaking up Groups*). *Suppose there exists an RGDD$(k_1, g; i \cdot g)$ and a* URD $(\{k_1, k_2\}; g)$ *with $r_{k_2} = j$, then there exists a* URD $(\{k_1, k_2\}; i \cdot g)$ *with $r_{k_2} = j$ and an IURD $(\{k_1, k_2\}; i \cdot g)$ with a hole of size g, $r_{k_1} = \frac{(i-1)\cdot g}{k_1-1}$ k_1-pcs, $r^\circ_{k_1} = \frac{g-1-(k_2-1)\cdot j}{k_1-1}$ holey (or partial) k_1-pcs, $r_{k_2} = 0$ k_2-pcs and $r^\circ_{k_2} = j$ holey k_2-pcs.*

Proof. Fill all groups of the RGDD with the URD to obtain the URD. Leave only one group empty to get the IURD. □

Construction 3.2 (*Generalized Frame Construction*). *Suppose there is a k_1-frame of type $T = \{t_i : i = 1, \ldots, n\}$, i.e. $v = \sum_{i=1}^n t_i$. If there exists an IURD $(\{k_1, k_2\}; t_i + s)$ with a hole of size s, $r_{k_1} = \frac{t_i}{k_1-1}$, $r^\circ_{k_1} = \frac{s-1-(k_2-1)\cdot j_2}{k_1-1}{}_1$, $r_{k_2} = 0$ and $r^\circ_{k_2} = j_2$ for $i = 1, \ldots, n-1$, then there exists an IURD $(\{k_1, k_2\}; v + s)$ with a hole of size t_n+s, $r_{k_1} = \frac{\sum_{i=1}^{n-1} t_i}{k_1-1}$, $r^\circ_{k_1} = \frac{t_n}{k_1-1} + \frac{s-1-(k_2-1)\cdot j_2}{k_1-1}$, $r_{k_2} = 0$ and $r^\circ_{k_2} = j_2$. If there exists a* URD $(\{k_1, k_2\}; t_n + s)$ *with $r_{k_2} = j_2$ and therefore $r_{k_1} = \frac{t_n}{k_1-1} + \frac{s-1-(k_2-1)\cdot j_2}{k_1-1}$, then a* URD $(\{k_1, k_2\}; v + s)$ *with $r_{k_2} = j_2$ exists.*

Proof. First, fill all holes, up to the last one with IURDs. This gives the IURD. The hole of the frame with size t_i has $\frac{t_i}{k_1-1}k_1$-pcs, which can be extended with the k_1-pcs from the IURD. The holey pcs from all this IURDs combine to holey pcs of the new IURD $(\{k_1, k_2\}; v + s)$ with a hole of size $t_n + s$. These are j_2 holey k_2-pcs and $\frac{s-1-(k_2-1)\cdot j_2}{k_1-1}{}_1$ holey k_1-pcs. $\frac{t_n}{k_1-1}$ holey k_1-pcs are from the hole of size t_n of the frame. Filling the last hole with the URD $(\{k_1, k_2\}; t_n + s)$ with $r_{k_2} = j_2$ results in the URD $(\{k_1, k_2\}; v + s)$ with $r_{k_2} = j_2$. □

Remark. If in Construction 3.2 all IURDs derive from Construction 3.1 with $r^\circ_{k_2} = j_2$ and if the given URD has $r_{k_2} = j_2$, then all additional conditions in Construction 3.2 are fulfilled. In this paper, almost all IURDs derive from Construction 3.1.

Construction 3.3 (*Weighting [6]*). *Let $(X, \boldsymbol{G}, \boldsymbol{B})$ be a GDD, and let $w : X \to Z^+ \cup 0$ be a weight function on X. Suppose that for each block $B \in \mathbf{B}$, there exists a K-frame of type $\{w(x) : x \in B\}$. Then there is a K-frame of type $\left\{\sum_{x\in G_i} w(x) : G_i \in \boldsymbol{G}\right\}$.*

Construction 3.4 (*Tripling [4]*). *If there exists a uniform K-frame of type $g_1^{u_1} \cdots g_s^{u_s}$ and for each $k \in K$ there exists an RTD$(3, k)$ (i.e. there are no blocks of size 2 or 6), then a uniform $\{\{3\} \cup K\}$-frame of type $(3g_1)^{u_1} \cdots (3g_s)^{u_s}$ exists.*

Proof. Expand each point of the K-frame three times. Place an RTD$(3, k)$ on each expanded block of a block of size k in such a way that one parallel class of blocks of size 3 is the expansion of each point. Remove this parallel class to obtain the design. □

Please note that for $k \neq 3$ the number of parallel classes with blocks of size k is the same in both frames. Now, a Wilson type construction is shown, where each point of a *master* design is expanded and the resulting large blocks are filled with so-called *ingredient* designs.

Theorem 3.5. *There exists a* URD $(\{3, 5\}; 60 \cdot t + 15)$ *with* $r_5 \leq 5 \cdot t + 1$ *for* $t \geq 1$, $t \notin \{2, 17, 23, 32\}$.

Proof. By Theorem 1.2 we can take as a master design an RPBD$(5; 20 \cdot t + 5)$ for $t \geq 1$, $t \notin \{2, 17, 23, 32\}$, which has $5 \cdot t + 1$ parallel classes. Expand all points of this master design three times. By Theorem 1.3 there exists an RGDD$(3, 3; 15)$, which is our first ingredient design. There exists a URD $(\{3, 5\}; 15)$ with $r_5 = 1$ by Theorem 1.9. We take one parallel class of blocks of size 3 (3-pc) as groups and get an RGDD$(\{3, 5\}, 3; 15)$ with $r_5 = 1$ as the second ingredient design. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way so that several uniform parallel classes are created. Each 5-pc of the master design results in zero or one new 5-pc, therefore it is $r_5 \leq 5 \cdot t + 1$. Lastly all expanded points form a 3-pc. □

Theorem 3.6. *If there exists a* URD $(\{3, 5\}; v_0)$ *with* $r_3 > 0$, $r_5 = m > 0$, *then there exists a* URD $(\{3, 5\}; v)$ *with* $r_5 = m$, *and an IURD* $(\{3, 5\}; v)$ *with* $r_5^{\circ} = m$ *and a hole of size* v_0, *for all* $v \equiv v_0 \pmod{2 \cdot v_0}$, $v \geq 3 \cdot v_0$.

Proof. Since there exists a URD $(\{3, 5\}; v_0)$ with $r_3 > 0$, $r_5 = m > 0$, it is $v_0 \equiv 15 \pmod{30}$ by Theorem 1.8. Therefore, there exists a 3-RGDD of type v_0^n for $n \equiv 1 \pmod 2$, $n \geq 3$ by Theorem 1.3. Filling the groups by the above URD $(\{3, 5\}; v_0)$ with $r_5 = m$, results in the desired URD. Without filling one group, we get the IURD. □

Corollary 3.7. *There exist a* URD $(\{3, 5\}; v)$ *with* $r_5 = 2, 3, 4, 5$, *and an IURD* $(\{3, 5\}; v)$ *with* $r_5^{\circ} = 2, 3, 4, 5$ *and a hole of size 45, for all* $v \equiv 45 \pmod{90}$, $v \geq 135$.

Proof. A URD $(\{3, 5\}; 45)$ with $r_5 = 2, 3, 4, 5$ exists by Lemma 2.7. Therefore, the assertion follows by Theorem 3.6. □

Corollary 3.8. *There exist a* URD $(\{3, 5\}; v)$ *with* $r_5 = 2, 3, 4, 5$, *and an IURD* $(\{3, 5\}; v)$ *with* $r_5^{\circ} = 2, 3, 4, 5$ *and a hole of size 75, for all* $v \equiv 75 \pmod{150}$, $v \geq 225$.

Proof. *A* URD $(\{3, 5\}; 75)$ with $r_5 = 2, 3, 4, 5$ exists by Theorem 3.5. Therefore, the assertion follows by Theorem 3.6. □

The following frame results are based on ideas, which in [4] are developed for URD$(\{3, 4\}; v)$ with $r_4 = 3$.

Lemma 3.9. *There exists a* $\{5, 6\}$*-GDD of type* $5^{4t+1}u^1$, $0 \leq u \leq 5 \cdot t$, *for all* $t \geq 1$, $t \notin \{2, 17, 23, 32\}$.

Proof. There exists a 5-RGDD of type 5^{4t+1} for $t \geq 1$, $t \notin \{2, 17, 23, 32\}$ with $5t$ parallel classes by Theorem 1.2. Completing u parallel classes, results in the desired design. □

Lemma 3.10. *There exists a* $\{3, 5\}$*-frame of type* $(6; 3^3)^5(12; 3^4 5^1)^1$.

Proof. It is well known that a TD$(4, 5)$ exists, therefore also an RTD$(3, 5)$ and that is equivalent to a 3-RGDD of type 5^3. Deleting one point results in a $\{3, 5\}$-frame of type $(2; 3^1)^5(4; 5^1)^1$. Construction 3.4 makes the desired frame. □

Lemma 3.11. *There exists a* $\{3, 5\}$*-frame of type* $(30; 3^{15})^{4t+1}(6u + 6r; 3^{3u+r}5^r)^1$ *for* $0 \leq r \leq u \leq 5t$, $t \geq 1$, $t \notin \{2, 17, 23, 32\}$.

Proof. Take the $\{5, 6\}$-GDD of type $5^{4t+1}u^1$ from Lemma 3.9. Assign r points from the group of size u weight 12 and assign all other points weight 6. There exist a 3-frame of type 6^5 and a 3-frame of type 6^6 by Theorem 1.4. By Lemma 3.10 there exists a $\{3, 5\}$-frame of type $(6; 3^3)^5(12; 3^4 5^1)^1$. Take these frames in the weighting Construction 3.3 to get the result. □

Theorem 3.12. *If there exists a uniform* $\{3, 5\}$*-frame of type* $(g_1; 3^{\frac{g_1}{2}})^s(g_2; 3^{\frac{g_2-4 \cdot r}{2}}5^r)^1$ *and* $w \equiv 3 \pmod 6$ *is such that* $g_1 + w \equiv 3 \pmod 6$, $2 \cdot w \leq g_1$, *then there exists an IURD*$(\{3, 5\}; g_1 \cdot s + g_2 + w)$ *with a hole of size* $g_2 + w$, $r_5^{\circ} = r$, $r_5 = 0$ *and* $r_3^{\circ} = \frac{w-1}{2}$. *If there exists a* URD$(\{3, 5\}; g_2 + w)$ *with* $r_5 = r$, *and therefore* $r_3 = \frac{g_2+w-1-4 \cdot r}{2}$, *then there exists a* URD$(\{3, 5\}; g_1 \cdot s + g_2 + w)$ *with* $r_5 = r$.

Proof. First, add w infinite points on the frame. Then, fill each group G of size g_1 together with the w infinite points with the resolvable incomplete design RIPBD$(3; g_1 + w)$ with a hole of size w and $r_3^{\circ} = (w-1)/2$, which exists by Theorem 1.5, in such a way that the hole is filled with the infinite points. Each such group G has $g_1/2$ frame-3-pcs, which can be extended with the $g_1/2$ parallel classes of the RIPBD$(3; g_1 + w)$ with a hole of size w. All $(w-1)/2$ holy 3-pcs are left over. The result is the IURD. The group G_2 of size g_2 together with the w infinite points will be filled with the URD$(\{3, 5\}; g_2 + w)$ with $r_5 = r$ and $r_3 = \frac{g_2+w-1-4 \cdot r}{2}$. The $(g_2 - 4r)/2$ frame-3-pcs and also the r frame-5-pcs can be extended with pcs from the given URD. There remain $(w-1)/2$ parallel classes of size 3 of the given URD. These join with the 3-pcs of the other groups to $(w-1)/2$ additional 3-pcs of the new URD. □

4. Results for URDs with exactly 3 parallel classes with blocks of size 5

We use the frame in Lemma 2.9 to construct an IURD and with it the URDs.

Lemma 4.1. *There exists an IURD* $(\{3,5\}; 30+15)$ *with a hole of size 15,* $r_3 = 15$, $r_5 = 0$, $r_3^\circ = 1$, $r_5^\circ = 3$.

Proof. Take the $\{3,5\}$-frame of type $(6; 3^3)^5(12; 5^3)^1$ from Lemma 2.9. Adjoin 3 infinite points to the frame and fill all groups of size 6 with a 3-RGDD of type 3^3, where the infinite points form one group. The 3 pcs of the RGDD extend the 3 holey pcs of the frame, therefore is $r_3 = 15$. Together all groups of the above RGDDs become one holey 3-pc, i.e. $r_3^\circ = 1$. The group of size 12 of the frame and the infinite points give the hole of size 15 and $r_5^\circ = 3$. □

Theorem 4.2. *There exists a* URD $(\{3,5\}; v)$ *with* $r_5 = 3$ *if, and only if,* $v \equiv 15 \pmod{30}$ *except for* $v = 15$*, and except possibly for* $v = 105$.

Proof (*Use of* Construction 3.2). Adjoin 15 infinite points to a 3-frame of type 30^n, $n \geq 4$ (Theorem 1.4) and fill all groups except one group together with the infinite points with an IURD $(\{3,5\}; 45)$ with $r_5^\circ = 3$ and a hole of size 15, which is given in Lemma 4.1, where the infinite points form the hole. For each group the 15 3-pcs from the frame can be extended with the 15 3-pcs from the IURD. This gives an IURD $(\{3,5\}; 30 \cdot n + 15)$ with a hole of size 45, $r_3 = 15 \cdot (n-1)$, $r_5 = 0$, $r_3^\circ = 1$, $r_5^\circ = 3$. Fill the last group together with the infinite points with a URD $(\{3,5\}; 45)$ with $r_5 = 3$ and $r_3 = 16$, which is given in Lemma 2.7. The 16 3-pcs from the URD complete all 15 3-pcs from the group of the frame and the only holy 3-pc from the IURD resulting in a URD$(\{3,5\}; 30 \cdot n + 15)$ with $r_5 = 3$. A URD$(\{3,5\}; 75)$ with $r_5 = 3$ is provided in Theorem 3.5. A URD $(\{3,5\}; 15)$ with $r_5 = 3$ cannot exist by the second necessary condition in Theorem 1.8. □

5. Results for URDs with exactly 2, 4 or 5 parallel classes with blocks of size 5

In this section all results are given for $r_5 = 2, 4, 5$ at the same time. All values, which are different for this values are written down separated with commas, for instance $u = 8, 6, 5$ in the proof of Lemma 5.3. The results are given for the four residue classes 15, 45, 75, 105 modulo 120, whereas 15 and 75 are given as 15 modulo 60.

Lemma 5.1. *There exists a* URD $(\{3,5\}; v)$ *with* $r_5 = 2, 4, 5$ *for all* $v \equiv 15 \pmod{60}$ *except for* $v = 15$.

Proof. By Theorem 3.5 there exists a URD $(\{3,5\}; 60 \cdot t + 15)$ with $r_5 = 2, 4, 5$ for $t \geq 1$, $t \notin \{2, 17, 23, 32\}$. For $v \in \{135, 1035, 1395, 1935\}$ there exists a URD $(\{3,5\}; v)$ with $r_5 = 2, 4, 5$ by Corollary 3.7. By the second necessary condition in Theorem 1.8 a URD $(\{3,5\}; 15)$ with $r_5 = 2, 4, 5$ cannot exist. □

Lemma 5.2. *There exists a* URD $(\{3,5\}; v)$ *with* $r_5 = 2, 4, 5$ *for all* $v \equiv 45 \pmod{120}$ *except possibly when* $v \in \{165, 285\}$.

Proof. By Corollary 3.7 there exists a URD$(\{3,5\}; 135)$ with $r_5 = 2, 4, 5$. Let $w = 15$, $g_1 = 30$, $s = 4t+1$, $g_2 = 120$, $u = 18, 16, 15$. Use this design, Lemma 3.11 and Theorem 3.12 to get a URD $(\{3,5\}; 30 \cdot (4t+1) + 135)$ with $r_5 = 2, 4, 5$ for $t \geq 4, 4, 3$, $t \notin \{2, 17, 23, 32\}$. For $v \in \{405, 2205, 2925, 4005\}$ there exists a URD $(\{3,5\}; v)$ with $r_5 = 2, 4, 5$ by Corollary 3.7. There exists a URD $(\{3,5\}; 45)$ with $r_5 = 2, 4, 5$ by Lemma 2.7. A URD $(\{3,5\}; 525)$ with $r_5 = 2, 4, 5$ exists by Corollary 3.8. □

Lemma 5.3. *There exists a* URD $(\{3,5\}; v)$ *with* $r_5 = 2, 4, 5$ *for all* $v \equiv 105 \pmod{120}$ *except possibly when* $v \in \{105, 345, 2145, 2865, 3945\}$*. Also, there exists an IURD* $(\{3,5\}; 465)$ *with* $r_5^\circ = 2, 4, 5$ *and a hole of size 75.*

Proof. Let $w = 15$, $g_1 = 30$, $s = 4t+1$, $g_2 = 60$, $u = 8, 6, 5$. Use a URD $(\{3,5\}; 75)$ with $r_5 = 2, 4, 5$ (see Theorem 3.5), Lemma 3.11 and Theorem 3.12 getting a URD $(\{3,5\}; 30 \cdot (4t+1) + 75)$ with $r_5 = 2, 4, 5$ for $t \geq 3, 3, 1$, $t \notin \{2, 17, 23, 32\}$. A URD $(\{3,5\}; 225)$ is given in Corollary 3.7. Without filling the hole of size g_2 of the frame, we get IURDs. Particularly for $t = 3$ an IURD $(\{3,5\}; 465)$ with $r_5^\circ = 2, 4, 5$ and a hole of size $g_2 + w = 75$ is obtained. □

Now, 3 from the undecided cases of Lemma 5.3 will be constructed with GDDs.

Lemma 5.4. *There exists a* URD $(\{3,5\}; 2145)$ *with* $r_5 = 2, 4, 5$.

Proof. Take a 4-GDD of type $39^4 51^1$, which exists by Theorem 1.1. Apply Construction 3.3 with weight 10 and 3-frames of type 10^4, which exists by Theorem 1.4. The result is a 3-frame of type $390^4 \cdot 510^1$. Take the IURD $(\{3,5\}; 390+75)$ with $r_5^\circ = 2, 4, 5$ and a hole of size 75, which is given in Lemma 5.3. Adjoin 75 infinite points to the frame and fill all groups of size 390 with this IURD, where the infinite points form the hole. Fill the group of size 510 together with the infinite points with a URD $(\{3,5\}; 585)$ with $r_5 = 2, 4, 5$, which is also given in Lemma 5.3. This results in the required design. □

Lemma 5.5. *There exists a* URD $(\{3,5\}; 2865)$ *with* $r_5 = 2, 4, 5$.

Proof. Take a 4-GDD of type $60^4 39^1$, which exists by Theorem 1.1. Apply Construction 3.3 with weight 10 and 3-frames of type 10^4, which exists by Theorem 1.4. The result is a 3-frame of type $600^4 \cdot 390^1$. By Corollary 3.8 there exists an IURD $(\{3,5\}; 600+75)$ with $r_5 = 2, 4, 5$ and a hole of size 75. Adjoin 75 infinite points to the frame and fill all groups of size 600 with this IURD, where the infinite points form the hole. Fill the group of size 390 together with the infinite points with a URD $(\{3,5\}; 465)$ with $r_5 = 2, 4, 5$, which is given in Lemma 5.3. This results in the required design. □

Lemma 5.6. *There exists a* URD $(\{3, 5\}; 3945)$ *with* $r_5 = 2, 4, 5$.

Proof. Take the 4-GDD of type $81^4 66^1$, which exists by Theorem 1.1. Apply Construction 3.3 with weight 10 and 3-frames of type 10^4, which exists by Theorem 1.4. The result is a 3-frame of type $810^4 \cdot 660^1$. By Corollary 3.7 there exists an IURD $(\{3, 5\}; 810 + 45)$ with $r_5 = 2, 4, 5$ and a hole of size 45. Adjoin 45 infinite points to the above frame and fill all groups of size 810 with this IURD, where the infinite points form the hole. Fill the group of size 660 together with the infinite points with a URD $(\{3, 5\}; 705)$ with $r_5 = 2, 4, 5$, which is given in Lemma 5.3. This results in the required design. □

All lemmas and theorems of the preceding two sections give our main theorem:

Theorem 5.7. *There exists a* URD $(\{3, 5\}; v)$ *with* $r_5 = 2, 3, 4, 5$ *if, and only if,* $v \equiv 15 \pmod{30}$ *except* $v = 15$, *and except possibly*

$$v = 105 \quad \text{for } r_5 = 3,$$
$$v \in \{105, 165, 285, 345\} \quad \text{for } r_5 = 2, 4, 5.$$

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