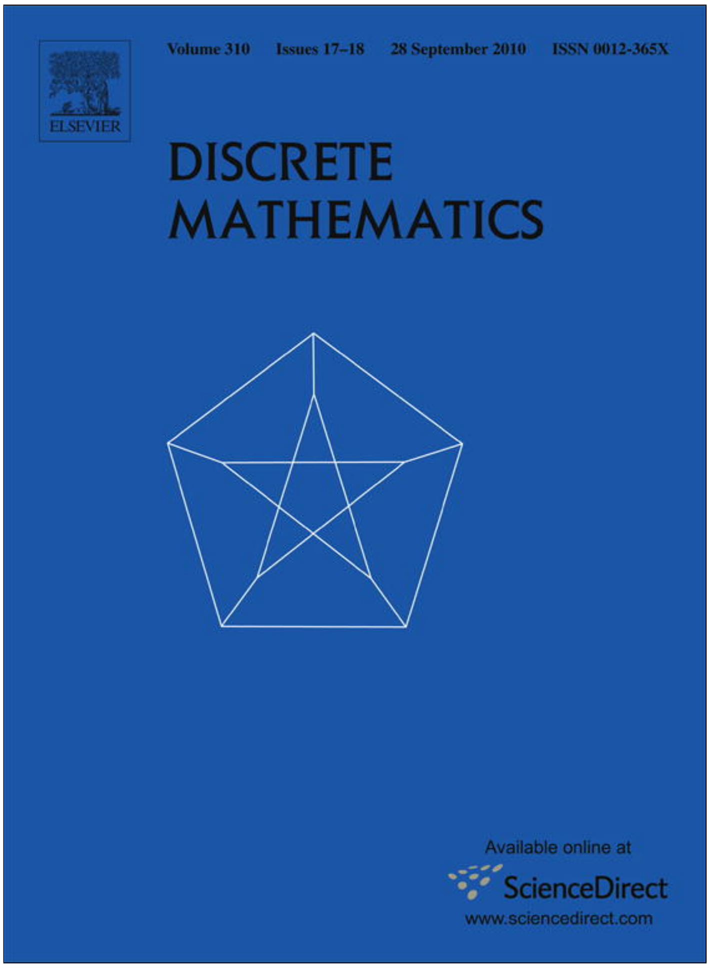
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Group divisible designs with block size four and group type $g^u m^1$ where g is a multiple of 8

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ABSTRACT

We determine, up to some possible exceptions, the spectrum for 4-GDDs of type $g^u m^1$, where g is a multiple of 8 until 48, g is a multiple of 24 until 144, respectively. These spectra are without exceptions for $g = 8, 16, 24, 48, 72, 96, 120$ and 144. Furthermore, we establish, up to a finite number of possible exceptions, the spectra for 4-GDDs of types $30^u m^1$ and $90^u m^1$. Finally, we provide nine $\{3, 5\}$-URDs which were the last possible exceptions in their classes.

1. Introduction

A *group divisible design* (GDD) with *index* λ is a triple $(X, \boldsymbol{G}, \boldsymbol{B})$, where X is a set of points, $\boldsymbol{G}$ is a partition of X into groups, and $\boldsymbol{B}$ is a collection of subsets of X called blocks such that any pair of distinct points from X occurs either in some group or in exactly λ blocks, but not both. A K-GDD$_\lambda$ of type $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ is a GDD in which every block has a size from the set K and in which there are u_i groups of size g_i, $i = 1, 2, \dots, s$. The notation is similar to [3,6]. If $\lambda = 1$, the index λ is omitted. If $K = \{k\}$ then the K-GDD$_\lambda$ is simply denoted k-GDD$_\lambda$.

Theorem 1.1 ([4]). *Let g and u be positive integers. Then there exists a* 4-GDD *of type g^u if, and only if, the conditions in Table 1 are satisfied.*

The necessary conditions for a 4-GDD of type $g^u m^1$ with $g, m > 0$ and $u \geq 4$ are summarized in Table 2.

Theorem 1.2 ([12]). *The necessary conditions of Table 2 for a 4-GDD of type $g^u m^1$ are sufficient for the minimum values of m, except that there is no 4-GDD of type $6^4 0^1$, but a 4-GDD of type $6^4 3^1$, and except possibly for the types $11^{12} 2^1$, $11^{17} 2^1$, $11^{21} 2^1$ and $11^{27} 5^1$. The necessary conditions of Table 2 for a 4-GDD of type $g^u m^1$ are sufficient for the maximum values of m, except that there is no 4-GDD of type $2^6 5^1$.*

Theorem 1.3 ([17,22]). *There exists a* 4-GDD *of type $g^4 m^1$ with $m > 0$ if, and only if, $g \equiv m \equiv 0 \pmod 3$ and $0 < m \leq 3g/2$.*

There exists a similar theorem for block size 5.

Theorem 1.4 ([1,13,22]). *There exists a* 5-GDD *of type $g^5 m^1$ with $m > 0$ if $g \equiv m \equiv 0 \pmod 4$ and $0 < m \leq 4g/3$, with the possible exceptions of $(g, m) = (12, 4)$ and $(12, 8)$.*

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Table 1
Existence of 4-GDDs of type g^u.

g	u	Necessary and sufficient conditions
$\equiv 1, 5 \pmod 6$	$\equiv 1, 4 \pmod{12}$	$u \geq 4$
$\equiv 2, 4 \pmod 6$	$\equiv 1 \pmod 3$	$u \geq 4$, $(g, u) \neq (2, 4)$
$\equiv 3 \pmod 6$	$\equiv 0, 1 \pmod 4$	$u \geq 4$
$\equiv 0 \pmod 6$	No constraint	$u \geq 4$, $(g, u) \neq (6, 4)$

Table 2
Necessary existence criteria for a 4-GDD of type $g^u m^1$ with $u \geq 4$.

g	u	m	$m_{\min}$	$m_{\max}$
$\equiv 0 \pmod 6$	No conditions	$\equiv 0 \pmod 3$	0	$g(u-1)/2$
$\equiv 1 \pmod 6$	$\equiv 0 \pmod{12}$	$\equiv 1 \pmod 3$	1	$(g(u-1)-3)/2$
	$\equiv 3 \pmod{12}$	$\equiv 1 \pmod 6$	1	$g(u-1)/2$
	$\equiv 9 \pmod{12}$	$\equiv 4 \pmod 6$	4	$g(u-1)/2$
$\equiv 2 \pmod 6$	$\equiv 0 \pmod 3$	$\equiv 2 \pmod 3$	2	$g(u-1)/2$
$\equiv 3 \pmod 6$	$\equiv 0 \pmod 4$	$\equiv 0 \pmod 3$	0	$(g(u-1)-3)/2$
	$\equiv 1 \pmod 4$	$\equiv 0 \pmod 6$	0	$g(u-1)/2$
	$\equiv 3 \pmod 4$	$\equiv 3 \pmod 6$	3	$g(u-1)/2$
$\equiv 4 \pmod 6$	$\equiv 0 \pmod 3$	$\equiv 1 \pmod 3$	1	$g(u-1)/2$
$\equiv 5 \pmod 6$	$\equiv 0 \pmod{12}$	$\equiv 2 \pmod 3$	2	$(g(u-1)-3)/2$
	$\equiv 3 \pmod{12}$	$\equiv 5 \pmod 6$	5	$g(u-1)/2$
	$\equiv 9 \pmod{12}$	$\equiv 2 \pmod 6$	2	$g(u-1)/2$

For some small values of g, an almost complete solution was found.

Theorem 1.5 ([23,18,14]).

1. *A 4-GDD of type $1^u m^1$ exists if, and only if, $u \geq 2m+1$ and either $m, u+m \equiv 1$ or $4 \pmod{12}$ or $m, u+m \equiv 7$ or $10 \pmod{12}$.*
2. *There exists a 4-GDD of type $2^u m^1$ for each $u \geq 6$, $u \equiv 0 \pmod 3$ and $m \equiv 2 \pmod 3$ with $2 \leq m \leq u-1$ except for $(u, m) = (6, 5)$ and possibly excepting $(u, m) \in \{(21, 17), (33, 23), (33, 29), (39, 35), (57, 44)\}$.*
3. *A 4-GDD of type $3^u m^1$ exists if, and only if, either $u \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 3$, $0 \leq m \leq (3(u-1)-3)/2$; or $u \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 6$, $0 \leq m \leq 3(u-1)/2$; or $u \equiv 3 \pmod 4$ and $m \equiv 3 \pmod 6$, $0 < m \leq 3(u-1)/2$.*
4. *There exists a 4-GDD of type $4^u m^1$ for each $u \geq 6$, $u \equiv 0 \pmod 3$ and $m \equiv 1 \pmod 3$ with $1 \leq m \leq 2(u-1)$.*
5. *A 4-GDD of type $5^u m^1$ exists if, and only if, either $u \equiv 3 \pmod{12}$ and $m \equiv 5 \pmod 6$, $5 \leq m \leq 5(u-1)/2$; or $u \equiv 9 \pmod{12}$ and $m \equiv 2 \pmod 6$, $2 \leq m \leq 5(u-1)/2$; or $u \equiv 0 \pmod{12}$ and $m \equiv 2 \pmod 3$, $2 \leq m \leq (5(u-1)-3)/2$.*
6. *There exists a 4-GDD of type $6^u m^1$ for each $u \geq 4$ and $m \equiv 0 \pmod 3$ with $0 \leq m \leq 3(u-1)$ except for $(u, m) = (4, 0)$ and possibly excepting $(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$.*
7. *There exists a 4-GDD of type $12^u m^1$ for each $u \geq 4$ and $m \equiv 0 \pmod 3$ with $0 \leq m \leq 6(u-1)$.*
8. *A 4-GDD of type $15^u m^1$ exists if, and only if, either $u \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 3$, $0 \leq m \leq (15(u-1)-3)/2$; or $u \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 6$, $0 \leq m \leq 15(u-1)/2$; or $u \equiv 3 \pmod 4$ and $m \equiv 3 \pmod 6$, $3 \leq m \leq 15(u-1)/2$.*

A *transversal design* $TD_\lambda(k, g)$, is equivalent to a k-GDD_λ of type g^k. That means, each block in a $TD_\lambda(k, g)$ contains a point from each group. If $\lambda = 1$, the index λ is omitted.

Theorem 1.6 ([2]). *A TD(k, g) exists in the following cases:*

1. *$k = 5$ and $g \geq 4$ and $g \notin \{6, 10\}$;*
2. *$k = 6$ and $g \geq 5$ and $g \notin \{6, 10, 14, 18, 22\}$;*
3. *$k = 7$ and $g \geq 7$ and $g \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$.*

In a K-GDD_λ, a *parallel class* is a set of blocks, which partitions X. If $\boldsymbol{B}$ can be partitioned into parallel classes, then the K-GDD_λ is called *resolvable* and denoted K-$RGDD_\lambda$. A parallel class is called *uniform* if it contains blocks of only one size k (k-pc). If all parallel classes of a K-$RGDD_\lambda$ are uniform, the design is called *uniformly resolvable*. The following theorem about RGDDs will be applied later.

Theorem 1.7 ([6–10,15,16,11,21,24,27,28,30,32]). *The necessary conditions for the existence of a k-RGDD of type h^n, namely, $n \geq k$, $h \cdot n \equiv 0 \pmod k$ and $h \cdot (n-1) \equiv 0 \pmod{k-1}$, are also sufficient for*

$k = 3$, except for $(h, n) \in \{(2, 3), (2, 6), (6, 3)\}$; and for

$k = 4$, except for $(h, n) \in \{(2, 4), (2, 10), (3, 4), (6, 4)\}$ and possibly excepting:

1. $h \equiv 2, 10 \pmod{12}$:
   $h = 2$ *and* $n \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 178, 184, 202, 214, 238, 250, 334\}$;
   $h = 10$ *and* $n \in \{4, 34, 52, 94\}$;
   $h \in [14, 454] \cup \{478, 502, 514, 526, 614, 626, 686\}$ *and* $n \in \{10, 70, 82\}$.
2. $h \equiv 6 \pmod{12}$: $h = 6$ *and* $n \in \{6, 68\}$; $h = 18$ *and* $n \in \{18, 38, 62\}$.
3. $h \equiv 9 \pmod{12}$: $h = 9$ *and* $n = 44$.
4. $h \equiv 0 \pmod{12}$: $h = 24$ *and* $n = 23$; $h = 36$ *and* $n \in \{11, 14, 15, 18, 23\}$.

A *resolvable transversal design* $\mathrm{RTD}_\lambda(k, g)$, is equivalent to a k-RGDD_λ of type g^k.

A *double group divisible design* (DGDD) is a quadruple $(X, \boldsymbol{G}, \boldsymbol{H}, \boldsymbol{B})$ where X is a set of points, $\boldsymbol{G}$ and $\boldsymbol{H}$ are partitions of X (into groups and holes, respectively) and $\boldsymbol{B}$ is a collection of subsets of X (blocks) such that

1. for each block $B \in \boldsymbol{B}$ and each $H \in \boldsymbol{H}$, $|B \cap H| \le 1$, and
2. any pair of distinct points from X which are not in the same hole occur in some group or in exactly λ blocks, but not both.

A K-DGDD of type $(g, h^v a^1)^u$ is a double group divisible design in which every block has a size from the set K and in which there are u groups of size g, each of which intersects each of the first v holes in h points and the last hole in a points. Thus, $g = hv + a$. For example, a k-DGDD of type $(g, h^v a^1)^k$ is a *holey transversal design* k-HTD of hole type $h^v a^1$ and is equivalent to a set of $k - 2$ holey MOLS of type $h^v a^1$.

Theorem 1.8 ([5,20]). *There exists a* 4-$\mathrm{DGDD}_\lambda(hv, h^v)^u$ *if, and only if,* $u, v \ge 4$ *and* $\lambda(u-1)(v-1)h \equiv 0 \pmod{3}$ *except for* $(u, h, v, \lambda) = (4, 1, 6, 1)$.

Construction 1.9 ([19]). *Suppose that there is a* 4-DGDD $(gu, g^u)^n$, *and a* 4-GDD *of type* $g^u m^1$, $g > 1$, $u \ge 4$, *where* m *is a non-negative integer. Then there is a* 4-GDD *of type* $(ng)^u m^1$.

Because there also exists a 4-GDD of type 4^4, we obtain:

Corollary 1.10. *Suppose there exists a* 4-GDD *of type* $g^u m^1$, $g > 1$, $u \ge 4$, *then there exist a* 4-GDD *of type* $(4g)^u m^1$ *and a* 4-GDD *of type* $(4g)^u (4m)^1$.

Theorem 1.11 ([34]). *Suppose* h *and* v *are positive integers and* a *is non-negative. Then there exists a* $4-\mathrm{HTD}$ *of hole type* $h^v a^1$ *if, and only if,* $v \ge 4$ *and* $0 \le a \le h(v-1)/2$ *except for* $(h, v, a) = (1, 5, 1)$ *or* $(1, 6, 0)$.

Construction 1.12 ([22,14]). *Suppose that there exists a* $4 - \mathrm{HTD}$ *of hole type* $h^v a^1$. *Then there exists a* $\{3, 4\}$-DGDD *of type* $(3hv, (3h)^v)^4$ *whose blocks of size* 3 *can be partitioned into* $9a$ *parallel classes.*

Theorem 1.13 ([33]). *Let* m, n *be two positive integers. Then there exists a* 4-GDD *of type* $(3m)^4(6m)^1(3n)^1$ *if, and only if,* $m \le n \le 2m$ *with four possible exceptions* $(m, n) = (3, 5), (4, 7), (6, 7),$ *or* $(6, 11)$.

Construction 1.14 ([1]). *Suppose a* $\mathrm{TD}(k+1, n)$ *exists. Let* $\delta = 0$ *or* 1, *and form a block of size* $n+\delta$ *on each group plus* δ *infinite points. Now delete a finite point, and use its blocks to define new groups. This gives a* $\{k+1, n+\delta\}$-GDD *of type* $k^n(n-1+\delta)^1$.

The concept of labeled resolvable designs is needed in order to get direct constructions for resolvable designs. This concept was introduced by Shen [29,31,30].

Let $(X, \boldsymbol{B})$ be a $\mathrm{(U)GDD}_\lambda(K, M; v)$ where $X = \{a_1, a_2, \ldots, a_v\}$ is totally ordered with ordering $a_1 < a_2 < \cdots < a_v$. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, it is supposed that $x_1 < x_2 < \cdots < x_k$. Let Z_λ be the group of residues modulo λ.

Let $\varphi : \boldsymbol{B} \to Z_\lambda^{\binom{k}{2}}$ be a mapping where for each $B = \{x_1, x_2, \ldots, x_k\} \in \boldsymbol{B}$, $k \in K$, $\varphi(B) = (\varphi(x_1, x_2), \ldots, \varphi(x_1, x_k), \varphi(x_2, x_3), \ldots, \varphi(x_2, x_k), \varphi(x_3, x_4), \ldots, \varphi(x_{k-1}, x_k))$, $\varphi(x_i, x_j) \in Z_\lambda$ for $1 \le i < j \le k$.

A $\mathrm{(U)GDD}_\lambda(K, M; v)$ is said to be a *labeled (uniform resolvable) group divisible design*, denoted $\mathrm{L(U)GDD}_\lambda(K, M; v)$, if there exists a mapping φ such that:

1. For each pair $\{x, y\} \subset X$ with $x < y$, contained in the blocks $B_1, B_2, \ldots, B_\lambda$, then $\varphi_i(x, y) \equiv \varphi_j(x, y)$ if, and only if, $i = j$ where the subscripts i and j denote the blocks to which the pair belongs, for $1 \le i, j \le \lambda$; and
2. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, $\varphi(x_r, x_s) + \varphi(x_s, x_t) \equiv \varphi(x_r, x_t) \pmod{\lambda}$, for $1 \le r < s < t \le k$.

Its blocks will be denoted in the following form:

$$(x_1 x_2 \ldots x_k; \varphi(x_1, x_2) \ldots \varphi(x_1, x_k)\varphi(x_2, x_3) \ldots \varphi(x_2, x_k)\varphi(x_3, x_4) \ldots \varphi(x_{k-1}, x_k)), \quad k \in K.$$

The above definition is firstly used in [25] and is a little bit more general than the definition by Shen [30] with $K = \{k\}$ or Shen and Wang [31] for transversal designs. The main application of the labeled designs is to blow up the point set of a given design with the following theorem [29] here extended for labeled (uniform resolvable) pairwise balanced designs.

Theorem 1.15 *([29,25,26]). If there exists a* K-L(U)GDD$_\lambda$ *of type* $g_1^{u_1}g_2^{u_2}\dots g_s^{u_s}$ *(with* r_k^L *classes of size k, for each* $k \in K$*), then there exists a* K-(U)GDD *of type* $(\lambda \cdot g_1)_{u_1}(\lambda \cdot g_2)_{u_2}\dots(\lambda \cdot g_s)_{u_s}$ *(with* $r_k = r_k^L$ *classes of size k, for each* $k \in K$*).*

A *uniformly resolvable design* URD $(\{3, 5\}; v)$ with $r_3 = a$ and $r_5 = b$ is a $\{3, 5\}$-GDD of type 1^v with all blocks of size 3 in a 3-pcs and all blocks of size 5 in b 5-pcs.

Theorem 1.16 *([26,27]). There exists a* URD $(\{3, 5\}; v)$ *with* $r_5 = 2, 3, 4, 5$ *if, and only if,* $v \equiv 15 \pmod{30}$ *except* $v = 15$, *and except possibly* $v \in \{165, 285, 345\}$ *for* $r_5 = 2, 4, 5$.

This Theorem 1.16 can be improved.

Theorem 1.17. *There exists a* URD $(\{3, 5\}; v)$ *with* $r_5 = 2, 3, 4, 5$ *if, and only if,* $v \equiv 15 \pmod{30}$ *except* $v = 15$.

Proof. A uniform $\{3, 5\}$-LRGDD$_\lambda$ of type 3^5 with $\lambda \in \{11, 19, 23\}$ and $r_5 \in \{2, 4, 5\}$ is given in the online resource [36]. Therefore, there exists a uniform $\{3, 5\}$-RGDD of type $(3\lambda)^5$ with $r_5 = 2, 4$ or 5 by Theorem 1.15. By filling all groups with an RPBD$(3; 3\lambda)$, we obtain a URD $(\{3, 5\}; 15\lambda)$ with $r_5 = 2, 4$ or 5, $\lambda \in \{11, 19, 23\}$. □

In the Section 2, direct constructions provide first results. In Section 3, all 4-GDDs of type $g^u m^1$ will be constructed, where g is some multiple of 8. In further sections additional results are described. It was not possible for me to always construct the 4-GDDs of type $g^u m^1$ when g is multiple of 8. Therefore, I have limited myself to g's less or equal 144. These results are important to construct all 4-GDDs of type $g^u m^1$ for $g \equiv 0 \pmod{24}$ or $g \equiv 0 \pmod{36}$ in a further paper.

2. Direct constructions and first results

All directly constructed designs were found computationally. Firstly, a $\{3, 4\}$-GDD$_\lambda$ of type g^u with all blocks of size 3 in m 3-pcs was searched by simulated annealing. If this was successful, we secondly tried to label this design by simulated annealing or integer programming.

Lemma 2.1. *There exists a* $\{3, 4\}$-LGDD$_4$ *of type* 2^6 *with all blocks of size* 3 *in* m 3-*pcs for* $m \in \{5, 11, 14, 17\}$, $\mathbf{G} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$.

Proof. The following designs were found computationally. A $\{3, 4\}$-LGDD$_4$ of type 2^6 with all blocks of size 3 in 5 3-pcs (each 3-pc is a row):

(1 8 10; 3 0 1), (2 6 11; 3 0 1), (3 5 12; 1 3 2), (4 7 9; 1 2 1),
(2 5 8; 1 3 2), (1 3 9; 2 3 1), (6 7 12; 2 0 2), (4 10 11; 3 3 0),
(2 4 6; 0 1 1), (5 7 10; 0 3 3), (1 9 12; 0 0 0), (3 8 11; 3 1 2),
(1 4 8; 1 2 1), (2 5 11; 3 3 0), (3 6 9; 0 0 0), (7 10 12; 0 3 3),
(1 7 11; 3 0 1), (3 5 8; 2 1 3), (4 6 9; 2 0 2), (2 10 12; 2 3 1),
(1 3 7 11; 0 0 2 0 2 2), (1 4 5 9; 2 3 1 1 3 2), (1 6 10 12; 0 3 3 3 3 0),
(1 5 8 10; 0 1 1 1 1 0), (3 6 7 10; 3 3 1 0 2 2), (1 3 6 12; 1 3 1 2 0 2),
(2 4 5 12; 1 0 0 3 3 0), (1 4 8 12; 0 0 2 0 2 2), (1 4 6 7; 3 2 1 3 2 3),
(1 5 9 11; 1 2 3 1 2 1), (2 3 9 12; 0 2 1 2 1 3), (4 5 8 12; 2 2 1 0 3 3),
(5 7 9 12; 1 0 1 3 0 1), (2 4 7 10; 3 3 0 0 1 1), (5 7 9 11; 3 3 3 0 0 0),
(6 8 9 12; 0 3 1 3 1 2), (1 6 7 11; 1 2 1 1 0 3), (2 6 8 10; 2 0 3 2 1 3),
(2 3 5 7; 2 2 0 0 2 2), (1 3 5 10; 3 2 2 3 3 0), (2 4 7 12; 2 1 2 3 0 1),
(3 6 8 11; 1 0 0 3 3 0), (2 6 8 9; 0 1 1 1 1 0), (3 8 10 12; 2 0 2 2 0 2),
(2 3 7 9; 1 2 0 1 3 2), (2 8 9 11; 2 3 1 1 3 2), (4 5 10 11; 0 2 1 2 1 3),
(4 6 10 11; 0 0 2 0 2 2), (4 8 9 11; 3 1 0 2 1 3), (2 3 10 11; 3 1 2 2 3 1).

A $\{3, 4\}$-LGDD$_4$ of type 2^6 with all blocks of size 3 in 11 3-pcs (each 3-pc is a row):

(2 3 12; 3 3 0), (4 5 8; 2 3 1), (1 6 9; 3 1 2), (7 10 11; 2 1 3),
(4 5 10; 1 2 1), (2 6 8; 2 1 3), (3 9 11; 2 0 2), (1 7 12; 2 2 0),
(1 5 12; 2 1 3), (2 3 7; 0 0 0), (4 6 9; 3 0 1), (8 10 11; 2 2 0),
(4 8 12; 1 3 2), (1 3 10; 2 2 0), (2 5 9; 1 1 0), (6 7 11; 0 2 2),
(1 3 5; 0 0 0), (4 6 11; 0 1 1), (7 9 12; 1 3 2), (2 8 10; 2 2 0),
(1 6 8; 1 1 0), (5 7 10; 3 3 0), (2 4 12; 2 0 2), (3 9 11; 0 1 1),
(1 5 9; 3 2 3), (2 7 10; 3 0 1), (3 8 12; 0 1 1), (4 6 11; 1 0 3),
(1 4 7; 1 1 0), (5 8 11; 2 2 0), (2 9 12; 0 1 1), (3 6 10; 2 2 0),
(1 8 9; 3 3 0), (4 10 12; 1 1 0), (2 5 11; 3 2 3), (3 6 7; 1 3 2),
(1 8 10; 2 1 3), (2 3 11; 1 3 2), (4 5 9; 0 2 2), (6 7 12; 1 3 2),
(1 5 11; 1 1 0), (2 6 12; 1 2 1), (3 7 9; 1 1 0), (4 8 10; 2 3 1),
(2 4 5 7; 3 2 2 3 3 0), (2 3 5 8; 2 0 3 2 1 3), (2 4 7 9; 0 1 3 1 3 2),
(1 4 7 11; 2 0 0 2 2 0), (5 7 9 11; 2 1 1 3 3 0), (3 6 8 9; 0 2 3 2 3 1),
(3 5 10 12; 1 3 2 2 1 3), (3 5 8 12; 3 3 3 0 0 0), (1 3 10 11; 3 0 2 1 3 2),

(2 6 10 11; 0 3 0 3 0 1), (5 7 10 12; 1 0 2 3 1 2), (1 4 9 12; 3 0 3 1 0 3),
(2 4 6 10; 1 3 1 2 0 2), (1 4 8 11; 0 0 3 0 3 3), (1 3 6 7; 1 0 3 3 2 3),
(1 6 10 12; 2 3 0 1 2 1), (2 8 9 11; 0 2 1 2 1 3), (6 8 9 12; 1 0 0 3 3 0).

A {3, 4}-LGDD$_4$ of type 2^6 with all blocks of size 3 in 14 3-pcs (each 3-pc is a row):
(1 5 11; 1 2 1), (2 6 7; 2 0 2), (3 9 12; 2 0 2), (4 8 10; 3 0 1),
(1 3 6; 3 3 0), (4 5 8; 2 2 0), (2 10 11; 0 0 0), (7 9 12; 0 0 0),
(1 6 12; 2 1 3), (2 3 7; 1 2 1), (5 9 11; 0 3 3), (4 8 10; 0 2 2),
(1 7 11; 0 3 3), (2 3 6; 0 3 3), (4 9 12; 3 0 1), (5 8 10; 3 3 0),
(1 3 8; 1 1 0), (2 6 9; 0 2 2), (4 5 11; 0 0 0), (7 10 12; 3 2 3),
(2 4 11; 2 1 3), (1 3 8; 2 0 2), (5 7 9; 3 2 3), (6 10 12; 0 0 0),
(1 4 5; 1 2 1), (2 3 12; 2 3 1), (6 7 10; 3 1 2), (8 9 11; 2 3 1),
(2 5 9; 2 3 1), (1 4 7; 0 1 1), (3 8 12; 3 3 0), (6 10 11; 2 0 2),
(2 3 10; 3 2 3), (5 8 12; 2 1 3), (4 6 11; 2 1 3), (1 7 9; 2 3 1),
(1 10 12; 2 3 1), (4 6 7; 0 0 0), (3 5 11; 1 3 2), (2 8 9; 3 0 1),
(1 5 10; 3 3 0), (2 4 9; 3 1 2), (3 8 11; 1 2 1), (6 7 12; 1 2 1),
(3 5 9; 2 1 3), (2 10 11; 1 2 1), (4 7 12; 3 2 3), (1 6 8; 0 3 3),
(1 5 12; 0 0 0), (2 8 10; 0 3 3), (4 6 9; 1 1 0), (3 7 11; 0 1 1),
(1 4 10; 2 1 3), (6 8 11; 1 1 0), (3 9 12; 3 2 3), (2 5 7; 0 1 1),
(3 5 7 10; 0 2 2 2 2 0), (1 4 10 12; 3 0 2 1 3 2), (2 6 8 12; 1 1 2 0 1 1),
(2 4 7 11; 1 3 3 2 2 0), (2 4 5 12; 0 3 1 3 1 2), (4 6 8 9; 3 1 0 2 1 3),
(2 5 8 12; 1 2 0 1 3 2), (1 3 6 9; 0 1 0 1 0 3), (1 7 9 11; 3 1 1 2 2 0),
(3 5 7 10; 3 3 0 0 1 1), (1 8 9 11; 2 2 0 0 2 2), (3 6 10 11; 2 1 0 3 2 3).

A {3, 4}-LGDD$_4$ of type 2^6 with all blocks of size 3 in 17 3-pcs (each 3-pc is a row):
(1 5 12; 1 0 3), (2 8 9; 1 1 0), (3 6 7; 0 1 1), (4 10 11; 1 1 0),
(2 3 11; 3 2 3), (1 4 6; 3 0 1), (8 10 12; 0 3 3), (5 7 9; 3 1 2),
(2 3 9; 1 3 2), (6 7 11; 0 0 0), (8 10 12; 2 2 0), (1 4 5; 1 0 3),
(3 6 9; 2 1 3), (4 5 8; 2 1 3), (1 10 11; 1 2 1), (2 7 12; 0 2 2),
(2 6 8; 0 2 2), (1 3 7; 0 0 0), (5 10 11; 2 1 3), (4 9 12; 2 3 1),
(1 8 10; 1 0 3), (2 4 12; 2 0 2), (3 5 7; 2 3 1), (6 9 11; 1 1 0),
(1 3 11; 3 0 1), (5 8 9; 0 2 2), (6 7 12; 3 2 3), (2 4 10; 1 0 3),
(1 8 9; 3 2 3), (3 10 12; 0 1 1), (2 5 11; 3 3 0), (4 6 7; 3 1 2),
(1 4 8; 0 2 2), (2 6 12; 3 3 0), (3 5 9; 0 3 3), (7 10 11; 0 2 2),
(1 5 12; 3 3 0), (3 8 10; 0 1 1), (4 7 11; 2 3 1), (2 6 9; 2 2 0),
(1 4 9; 2 3 1), (6 8 11; 0 2 2), (3 5 12; 3 0 1), (2 7 10; 3 2 3),
(1 7 9; 1 0 3), (2 4 12; 0 1 1), (3 6 10; 1 3 2), (5 8 11; 1 2 1),
(1 6 12; 3 2 3), (2 7 10; 1 3 2), (3 8 11; 3 2 3), (4 5 9; 0 0 0),
(1 6 10; 1 2 1), (3 9 12; 0 2 2), (2 5 7; 0 2 2), (4 8 11; 0 0 0),
(1 6 11; 2 1 3), (4 5 10; 1 0 3), (2 3 8; 2 3 1), (7 9 12; 1 1 0),
(2 9 11; 0 1 1), (1 3 7; 1 3 2), (4 6 8; 0 3 3), (5 10 12; 0 2 2),
(1 9 11; 1 3 2), (2 5 8; 2 0 2), (3 6 10; 3 2 3), (4 7 12; 0 0 0),
(4 7 9 11; 3 3 2 0 3 3), (1 3 8 12; 2 0 1 2 3 1), (1 5 7 10; 2 2 3 0 1 1),
(2 4 6 10; 3 1 1 2 2 0), (2 3 5 11; 0 1 0 1 0 3), (6 8 9 12; 1 2 1 1 0 3). □

Lemma 2.2. *There exists a* 4-GDD *of type* 8^6m^1 *for* $m \equiv 2 \pmod 3$, $2 \leq m \leq 20$.

Proof. There exists a 4-GDD of type 8^6m^1 for $m \in \{2, 20\}$ by Theorem 1.2. There exists a 4-GDD of type $8^7 \equiv 8^68^1$ by Theorem 1.1. There exists a {3, 4}-LGDD$_4$ of type 2^6 with all blocks of size 3 in m 3-pc, $\mathbf{G} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$ for $m \in \{5, 11, 14, 17\}$ by Lemma 2.1, which results in a {3, 4}-GDD of type 8^6 with all blocks of size 3 in m 3-pc for $m \in \{5, 11, 14, 17\}$ by Theorem 1.15. Completing all 3-pcs, we obtain the desired designs. □

Lemma 2.3. *There exists a* {3, 4}-LGDD$_8$ *of type* 1^9 *with all blocks of size* 3 *in* m_1 3-*pc,* $m_1 \in \{14, 26\}$. *There exists a* {3, 4}-LGDD$_4$ *of type* 2^9 *with all blocks of size* 3 *in* m_2 3-*pc, for* $m_2 \in \{11, 17, 23, 29\}$; $\mathbf{G} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}, \{13, 14\}, \{15, 16\}, \{17, 18\}\}$.

Proof. All these designs can be found in the online resource [36]. □

Lemma 2.4. *There exists a* {3, 4}-LGDD$_8$ *of type* 2^6 *with all blocks of size* 3 *in* m 3-*pcs for* $m \in \{13, 19, 22, 25, 31, 34, 37\}$, $\mathbf{G} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$.

Proof. All these designs can be found in the online resource [36]. □

Lemma 2.5. *There exists a* {3, 4}-LGDD$_{16}$ *of type* 1^9 *with all blocks of size* 3 *in* m 3-*pcs for* $m \in \{22, 34, 46, 58\}$.

Proof. All these designs can be found in the online resource [36]. □

Lemma 2.6. *There exists a* {3, 4}-LGDD$_8$ *of type* 2^9 *with all blocks of size* 3 *in* m 3-*pcs for* $m \in \{19, 25, 31, 37, 43, 49, 55, 61\}$, $\mathbf{G} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}, \{13, 14\}, \{15, 16\}, \{17, 18\}\}$.

Proof. All these designs can be found in the online resource [36]. □

3. Above 4-GDDs of type $g^u m^1$ for $g = 8, 16, 24, 32, 48, 72, 96$ and 144

Theorem 3.1. *A* 4-GDD *of type* $48^u m^1$ *exists if, and only if,* $u \geq 4$, $m \equiv 0 \pmod 3$ *and* $0 \leq m \leq 24(u-1)$.

Proof. There exists a TD$(5, u)$ for $u \geq 4$ and $u \notin \{6, 10\}$ by Theorem 1.6. Remove a point and use this point to redefine the groups. Complete all groups of size u with a new point. This gives a $\{5, u+1\}$-GDD of type $4^u u^1$ as our master design. There exist 4-GDDs of types 12^5, $12^4 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 18$, and 4-GDDs of types $12^u a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \leq a_0 \leq 6(u-1)$ by Theorem 1.5. We give every point in a group of size 4 in the master design weight 12. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain weights as a and the new point as a_0. The result is a 4-GDD of type $48^u m^1$ for $u \geq 4$, $u \notin \{6, 10\}$ and $m \equiv 0 \pmod 3$, $0 \leq m \leq 18(u-1) + 6(u-1) = 24(u-1)$.

By Theorem 1.6 there exists a TD$(7, 8)$. This is a 7-GDD of type $8^7 = 8^6 8^1$ which we use as our master design. There exists a 4-GDD of type $6^6 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 15$ by Theorem 1.5. In the last group of the master design the points obtain appropriate weights. All other points weight 6. The result is a 4-GDD of type $48^6 m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 120$.

By Theorem 1.7 there exists a 4-RGDDof type 4^{10}. Completing all parallel classes, results in a 5-GDD of type $4^{10} 12^1$ which we take as our master design. There exist 4-GDDs of types 12^5, $12^4 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 18$ by Theorem 1.5. We give every point in a group of size 4 in the master design weight 12. The points in the group of size 12 obtain appropriate weights. The result is a 4-GDD of type $48^{10} m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 12 \cdot 18 = 216 = 24 \cdot 9$. □

Lemma 3.2. *There exists a* 4-GDD *of type* $72^u m^1$ *for* $u \geq 4$, $m \equiv 0 \pmod 3$ *with* $12(u-1) \leq m \leq 36(u-1)$.

Proof. There exists a 4-RGDD of type 12^u, $u \geq 4$ by Theorem 1.7. Completing the parallel classes results in a 5-GDD of type $12^u (4(u-1))^1$ our master design. There exists a 4-GDD of type $6^4 a^1$, $a \equiv 0 \pmod 3$, $3 \leq a \leq 9$ by Theorem 1.5. In the last group of the master design the points obtain appropriate weights from a. All other points weight 6. The result is a 4-GDD of type $72^u m^1$, $m \equiv 0 \pmod 3$ and $12(u-1) \leq m \leq 36(u-1)$. □

Theorem 3.3. *A* 4-GDD *of type* $72^u m^1$ *exists if, and only if,* $u \geq 4$, $m \equiv 0 \pmod 3$ *and* $0 \leq m \leq 36(u-1)$.

Proof. There exists a 4-GDD of type $72^4 m^1$ for $m \equiv 0 \pmod 3$ with $0 \leq m \leq 108$ by Theorem 1.3.

There exists a TD$(7, 7)$ by Theorem 1.6, which gives a $\{7, 8\}$-GDD of type $6^7 7^1$ by Construction 1.14. We remove all points from two groups of size 6 and get a $\{5, 6, 7, 8\}$-GDD of type $6^5 7^1$ as our master design. There exist 4-GDDs of types $12^4 a^1$, $12^5 a^1$, $12^6 a^1$, $12^7 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 18$ by Theorem 1.5, which we apply as ingredient designs. The points in the last group obtain appropriate weights. All other points obtain weight 12. The result is a 4-GDD of type $72^5 m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 18 \cdot 7 = 126$.

Let $M_7 = \{6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$. Then there exists a TD$(7, u)$ for $u \geq 7$ with $u \notin M_7$ by Theorem 1.6 and we obtain a $\{7, u+1\}$-GDD of type $6^u u^1$ by Construction 1.14 as our master design. There exist 4-GDDs of types 12^7, $12^6 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 30$, $12^u a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \leq a_0 \leq 6(u-1)$ by Theorem 1.5. We give every point in a group of size 6 in the master design weight 12. The points in the group of size u obtain appropriate weights. The result is a 4-GDD of type $72^u m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 30(u-1) + 6(u-1) = 36(u-1)$.

There exists a TD$(7, u+1)$ for $u \in \{M_7 \setminus \{14\}\}$ by Theorem 1.6 and we obtain a $\{7, u+2\}$-GDD of type $6^{u+1}(u+1)^1$ by Construction 1.14. Deleting all points from one group of size 6 we get a $\{6, 7, u+1, u+2\}$-GDD of type $6^u (u+1)^1$ as our master design. There exist 4-GDDs of types $12^5 a^1$, $12^6 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 24$, $12^u a_0^1$, $12^{u+1} a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \leq a_0 \leq 6(u-1)$ by Theorem 1.5. We give every point in a group of size 6 in the master design weight 12. The points in the group of size $u+1$ obtain appropriate weights. The result is a 4-GDD of type $72^u m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 24u + 6(u-1) = 30(u-1) + 24$.

Now the last case $u = 14$. There exists a TD$(7, 16)$ by Theorem 1.6, which gives a $\{7, 17\}$-GDD of type $6^{16} 16^1$ by Construction 1.14. We remove all points from two groups of size 6 and get a $\{5, 6, 7, 15, 16, 17\}$-GDD of type $6^{14} 16^1$ as our master design. There exist 4-GDDs of types $12^4 a^1$, $12^5 a^1$, $12^6 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 18$, $12^{14} a_0^1$, $12^{15} a_0^1$, $12^{16} a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \leq a_0 \leq 6(14-1)$ by Theorem 1.5, which we apply as ingredient designs. The points in the last group obtain appropriate weights. All other points obtain weight 12. The result is a 4-GDD of type $72^{14} m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 18 \cdot 15 + 78 = 26 \cdot 13 + 10$. The assertion follows with Lemma 3.2. □

Theorem 3.4. *A* 4-GDD *of type* $144^u m^1$ *exists if, and only if,* $u \geq 4$, $m \equiv 0 \pmod 3$ *and* $0 \leq m \leq 72(u-1)$.

Proof. There exists a 4-RGDD of type 12^u for $u \geq 4$ by Theorem 1.7. Completing all pcs results in a 5-GDD of type $12^u (4(u-1))^1$ which is our master design. There exists a 4-GDD of type $12^4 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 18$. The points in the last group obtain appropriate weights. All other points obtain weight 12. The result is a 4-GDD of type $144^u m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 18 \cdot 4(u-1) = 72(u-1)$. □

Theorem 3.5. *There exists a* 4-GDD *of type* $16^u m^1$ *if, and only if,* $(u, m) = (3, 16)$ *or* $u \geq 6$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ *and* $1 \leq m \leq 8(u-1)$.

Proof. There exists a 4-GDD of type $48^{\hat{u}}m^1$ for $\hat{u} \geq 4$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 24(\hat{u}-1)$ by Theorem 3.1. Adjoin 16 infinite points and fill all groups of size 48 with a 4-GDD of type 16^4 (Theorem 1.1), where the infinite points form a group. This gives a 4-GDD of type $16^{3\hat{u}}(m+16)^1$ for $\hat{u} \geq 4$, $m \equiv 0 \pmod 3$ and $16 \leq m+16 \leq 24(\hat{u}-1)+16 = 8(3\hat{u}-1)$.

There exists a 4-GDD of type $4^u m^1$ for each $u \geq 6$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ and $1 \leq m \leq 2(u-1)$ by Theorem 1.5. Therefore, there exist a 4-GDD of type $16^u m^1$ and a 4-GDD of type $16^u(4m)^1$ for each $u \geq 6$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ and $1 \leq m \leq 2(u-1)$ by Corollary 1.10.

The designs of Lemma 2.4 in Appendix result in 4-GDDs of type $16^6 m^1$, $m \in \{13, 19, 22, 25, 31, 34, 37\}$ by Theorem 1.15 and completing all 3-pcs. The designs of Lemma 2.5 result in 4-GDDs of type $16^9 m^1$, $m \in \{22, 34, 46, 58\}$ by Theorem 1.15 and completing all 3-pcs. The designs of Lemma 2.6 result in 4-GDDs of type $16^9 m^1$, $m \in \{19, 25, 31, 37, 43, 49, 55, 61\}$ by Theorem 1.15 and completing all 3-pcs. □

Lemma 3.6. *A* 4-GDD *of type* $24^6 m^1$ *exists if, and only if,* $m \equiv 0 \pmod 3$ *and* $0 \leq m \leq 12 \cdot 5$.

Proof. There exists a 4-HTD of hole type $2^6 a^1$, $0 \leq a \leq 5$ by Theorem 1.11 and therefore a $\{3, 4\}$-DGDD of type $(36, 6^6)^4$ whose blocks of size 3 can be partitioned into $9a$ parallel classes by Construction 1.12. Adjoin $9a$ infinite points to complete the 3-pcs and then adjoin a further m ideal points, filling in 4-GDDs of type $6^6 m^1$, $m \equiv 0 \pmod 3$, $0 \leq m \leq 15$ coming from Theorem 1.5 to obtain a 4-GDD of type $24^6(9a+m)^1$, $0 \leq a \leq 5$, $m \equiv 0 \pmod 3$, $0 \leq m \leq 15$. □

Lemma 3.7. *A* 4-GDD *of type* $24^{10} m^1$ *exists if, and only if,* $m \equiv 0 \pmod 3$ *with* $0 \leq m \leq 12 \cdot 9$.

Proof. For $m \equiv 0 \pmod 3$, $0 \leq m \leq 27$ there exists a 4-GDD of type $6^{10} m^1$ by Theorem 1.5 and therefore, also a 4-GDD of type $24^{10} m^1$ by Corollary 1.10.

There exists a 5-GDD of type 4^{11} by [35]. Filling in 4-GDDs of types $6^4 3^1$, $6^4 6^1$, $6^4 9^1$, which are given in Theorem 1.5, we obtain a 4-GDD of type $24^{10} m^1$ for $m \equiv 0 \pmod 3$, $12 \leq m \leq 36$.

Completing a 4-RGDD of type 4^{10} (Theorem 1.7) results in a 5-GDD of type $4^{10} 12^1$. Filling in 4-GDDs of types $6^4 3^1$, $6^4 6^1$, $6^4 9^1$, which are given in Theorem 1.5, we obtain a 4-GDD of type $24^{10} m^1$ for $m \equiv 0 \pmod 3$, $36 \leq m \leq 12 \cdot 9$. □

Theorem 3.8. *There exists a* 4-GDD *of type* $24^u m^1$ *if, and only if,* $u \geq 4$ *and* $m \equiv 0 \pmod 3$ *with* $0 \leq m \leq 12(u-1)$.

Proof. A 4-GDD of type $24^4 \equiv 24^3 24^1$ exists by Theorem 1.1. For $u \in \{4, 6, 10\}$ the results are given in Theorem 1.3, Lemmas 3.6 and 3.7. There exists a TD$(5, u)$ for $u \geq 5$, $u \notin \{6, 10\}$ by Theorem 1.6. Remove a point and use this point to redefine the groups. Complete the groups of size u with a new point. This gives a $\{5, u+1\}$-GDD of type $4^u u^1$ as our master design. There exist 4-GDDs of types $6^4 3^1$, $6^4 6^1$, $6^4 9^1$, and 4-GDDs of type $6^u a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \leq a_0 \leq 3(u-1)$ by Theorem 1.5 with some possible exceptions for a_0. We give every point in a group of size 4 in the master design weight 6. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain 3, 6 or 9 as weight and the new point weights as a_0. The result is a 4-GDD of type $24^u m^1$ for $u \geq 4$, $u \notin \{6, 10\}$ and $m \equiv 0 \pmod 3$, $3(u-1) \leq m \leq 9(u-1) + 3(u-1) = 12(u-1)$. The possible exceptions for a_0 are not on the lower or upper limit. Therefore, they are not needed.

There exists a 4-GDD of type $6^u m^1$ for each $u \geq 4$ and $m \equiv 0 \pmod 3$ with $0 \leq m \leq 3(u-1)$ except for $(u, m) = (4, 0)$ and possibly excepting $(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$ by Theorem 1.5. By Corollary 1.10 there exists a 4-GDD of type $24^u m^1$ under the same conditions.

There exists a TD$(7, u)$, $u \in \{7, 13, 19\}$ by Theorem 1.6. Delete one point and use this point to redefine the groups. This gives a $\{7, u\}$-GDD of type $6^u(u-1)^1$, our master design. There exist 4-GDDs of types $4^6 1^1$, $4^6 4^1$, $4^6 7^1$, $4^6 10^1$ by Theorem 1.5 and 4^u, $u \in \{7, 13, 19\}$ by Theorem 1.1, which we apply as ingredient designs. The points in the last group obtain appropriate weights. All other points obtain weight 4. The result is a 4-GDD of type $24^u m^1$ for $m \equiv 0 \pmod 3$, $(u-1) \leq m \leq 10(u-1)$, $u \in \{7, 13, 19\}$.

By Theorem 1.1 there exists a 4-GDD of type $24^{12} \equiv 24^{11} 24^1$. There exist a 4-HTD of hole type $2^{11} a^1$, $0 \leq a \leq 10$ by Theorem 1.11 and therefore a $\{3, 4\}$-DGDD of type $(66, 6^{11})^4$ whose blocks of size 3 can be partitioned into $9a$ parallel classes by Construction 1.12. Adjoin $9a$ infinite points to complete the 3-pcs and then adjoin a further m ideal points, filling in 4-GDDs of type $6^{11} m^1$, $m \equiv 0 \pmod 3$, $0 \leq m \leq 30$, $m \notin \{21, 24, 27\}$ coming from Theorem 1.5 to obtain a 4-GDD of type $24^{11}(9+12)^1$ and a 4-GDD of type $24^{11}(9+18)^1$. By Theorem 1.4 there exists a 5-GDD of type $32^5 8^1 \equiv 32^4 8^1 32^1$. Truncating the last group appropriately, and filling the blocks with 4-GDDs of types 3^4 and 3^5, results in a 4-GDD of type $96^4 24^1 m^1$ for $m \equiv 0 \pmod 3$, $0 \leq m \leq 96$. There exists a 4-GDD of type 24^4 by Theorem 1.1. Filling all groups of size 96 with the above design results in a 4-GDD of type $24^{17} m^1$ for $m \equiv 0 \pmod 3$, $0 \leq m \leq 96$.

By Theorem 1.4 there exists a 5-GDD of type $40^5 24^1 \equiv 40^4 24^1 40^1$. Truncating the last group appropriately, and filling the blocks with 4-GDDs of types 3^4 and 3^5, results in a 4-GDD of type $120^4 72^1 m^1$ for $m \equiv 0 \pmod 3$, $0 \leq m \leq 120$. Adjoin 24 infinite points and fill all groups of size 120 with a 4-GDD of type 24^6 and the group of size 72 with a 4-GDD of type 24^4, where the infinite points form a group. The result is a 4-GDD of type $24^{23}(m+24)^1$ for $m \equiv 0 \pmod 3$, $0 \leq m \leq 120$. □

Lemma 3.9. *A* 4-GDD *of type* $8^9 m^1$ *exists if, and only if,* $m \equiv 2 \pmod 3$ *and* $2 \leq m \leq 32$.

Proof. There exists a 4-GDD of type 2^9m^1 for $m \in \{2, 5, 8\}$ by Theorem 1.5, and therefore, a 4-GDD of type 8^9m^1 for $m \in \{2, 5, 8, 20, 32\}$ by Corollary 1.10. The labeled designs of Lemma 2.3 result in 4-GDDs of type 8^9m^1 for $m \in \{11, 14, 17, 23, 26, 29\}$ by Theorem 1.15. □

Theorem 3.10. *There exists a* 4-GDD *of type* 8^um^1 *if, and only if,* $(u, m) = (3, 8)$ *or* $u \ge 6$, $u \equiv 0 \pmod 3$ *and* $m \equiv 2 \pmod 3$ *with* $2 \le m \le 4(u-1)$.

Proof. A 4-GDD of type $8^4 \equiv 8^38^1$ exists by Theorem 1.1. For $u \in \{6, 9\}$ the results are contained in Lemmas 2.2 and 3.9.

There exists a 4-GDD of type $24^{u_0}m_0^1$ for each $u_0 \ge 4$ and $m_0 \equiv 0 \pmod 3$ with $0 \le m_0 \le 12(u_0 - 1)$ by Theorem 3.8. Adjoin eight infinite points and fill all groups of size 24 with a 4-GDD of type 8^4, where the infinite points form a group. With $u = 3u_0$ and $m = m_0 + 8$ we obtain a 4-GDD of type 8^um^1 for each $u \ge 12$, $u \equiv 0 \pmod 3$ and $m \equiv 2 \pmod 3$ with $8 \le m \le 12(u_0 - 1) + 8 = 4(u - 1)$.

There exists a 4-GDD of type 8^u2^1 for $u \ge 12$, $u \equiv 0 \pmod 3$ by Theorem 1.2. By Theorem 1.5 there exists a 4-GDD of type 2^u5^1 for $u \ge 12$, $u \equiv 0 \pmod 3$. Therefore, there exists a 4-GDD of type 8^u5^1 for $u \ge 12$, $u \equiv 0 \pmod 3$ by Corollary 1.10. □

Theorem 3.11. *A* 4-GDD *of type* 96^um^1 *exists if, and only if,* $u \ge 4$, $m \equiv 0 \pmod 3$ *and* $0 \le m \le 48(u-1)$.

Proof. There exists a TD(5, u) for $u \ge 4$ and $u \notin \{6, 10\}$ by Theorem 1.6. Remove a point and use this point to redefine the groups. Complete all groups of size u with a new point. This gives a $\{5, u + 1\}$-GDD of type 4^uu^1 as our master design.

There exist 4-GDDs of types 24^5, 24^4a^1, $a \equiv 0 \pmod 3$, $0 \le a \le 36$, and a 4-GDD of type $24^ua_0^1$ if, and only if, $u \ge 4$ and $a_0 \equiv 0 \pmod 3$ with $0 \le a_0 \le 12(u-1)$ by Theorem 3.8.

We give every point in a group of size 4 in the master design weight 24. The points in the group of size u obtain appropriate weights. The $u - 1$ "old" points obtain weights as a and the new point as a_0. The result is a 4-GDD of type 96^um^1 for $u \ge 4$, $u \notin \{6, 10\}$ and $m \equiv 0 \pmod 3$, $0 \le m \le 36(u-1) + 12(u-1) = 48(u-1)$.

By Theorem 1.6 there exists a TD(7, 8). This is a 7-GDD of type $8^7 = 8^68^1$ which we use as our master design. There exists a 4-GDD of type 12^6a^1, $a \equiv 0 \pmod 3$, $0 \le a \le 30$ by Theorem 1.5. In the last group of the master design the points obtain appropriate weights. All other points weight 12. The result is a 4-GDD of type 96^6m^1, $m \equiv 0 \pmod 3$ and $0 \le m \le 240$.

By Theorem 1.7 there exists a 4-RGDDof type 4^{10}. Completing all parallel classes results in a 5-GDDof type $4^{10}12^1$ which we take as our master design. There exist 4-GDDs of types 24^5, 24^4a^1, $a \equiv 0 \pmod 3$, $0 \le a \le 36$ by Theorem 3.8. We give every point in a group of size 4 in the master design weight 24. The points in the group of size 12 obtain appropriate weights. The result is a 4-GDD of type $96^{10}m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 12 \cdot 36 = 48 \cdot 9$. □

Theorem 3.12. *There exists a* 4-GDD *of type* 32^um^1 *if, and only if,* $u \ge 4$, $u \equiv 0 \pmod 3$, $m \equiv 2 \pmod 3$ *and* $2 \le m \le 16(u-1)$, *possibly excepting* $u = 9$.

Proof. There exists a 4-GDD of type $96^{\hat{u}}m^1$ for $\hat{u} \ge 4$, $m \equiv 0 \pmod 3$ and $0 \le m \le 48(\hat{u} - 1)$ by Theorem 3.11. Adjoin 32 infinite points and fill all groups of size 96 with a 4-GDD of type 32^4 (Theorem 1.1), where the infinite points form a group. This gives a 4-GDD of type $32^{3\hat{u}}(m + 32)^1$ for $\hat{u} \ge 4$, $m \equiv 0 \pmod 3$ and $32 \le m + 32 \le 48(\hat{u} - 1) + 32 = 16(3\hat{u} - 1)$.

By Theorem 3.10 there exists a 4-GDD of type 8^um^1 for each $u \ge 6$, $u \equiv 0 \pmod 3$, $m \equiv 2 \pmod 3$ and $2 \le m \le 4(u-1)$. Therefore, there exists a 4-GDD of type 32^um^1 for each $u \ge 6$, $u \equiv 0 \pmod 3$, $m \equiv 2 \pmod 3$ and $2 \le m \le 4(u-1)$ by Corollary 1.10.

By Theorem 1.6 there exists a TD(7, 8). This is a 7-GDD of type $8^7 = 8^68^1$ which we use as our master design. There exist a 4-GDD of type 4^6a^1, $a \equiv 1 \pmod 3$, $1 \le a \le 10$ by Theorem 1.5. In the last group of the master design the points obtain appropriate weights as a. All other points weight 4. The result is a 4-GDD of type 32^6m^1, $m \equiv 2 \pmod 3$ and $8 \le m \le 10 \cdot 8 = 80 = 16(6-1)$. □

4. Above 4-GDDs of type 30^um^1

Let $M_6 = \{6, 10, 14, 18, 22\}$. Then there exists a TD(6, u) for $u \ge 5$ with $u \notin M_6$ by Theorem 1.6.

Lemma 4.1. *There exists a* 4-GDD *of type* 30^um^1 *for* $u \ge 4$, $u \notin M_6$ *and* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 15(u-1)$.

Proof. A 4-GDD of type 30^4m^1 for $u \ge 4$ and $m \equiv 0 \pmod 3$ with $0 \le m \le 45$ exists by Theorem 1.3. Let be $u \ge 5$, $u \notin M_6$ then there exists a TD(6, u), and, therefore, there exists a $\{6, u + 1\}$-GDD of type 5^uu^1 by Construction 1.14, which is our master design. There exist 4-GDDs of types 6^5, 6^5a^1, $a \equiv 0 \pmod 3$, $0 \le a \le 12$, and 4-GDDs of type $6^ua_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \le a_0 \le 3(u-1)$ by Theorem 1.5 with some possible exceptions for a_0. We give every point in a group of size 5 in the master design weight 6. The points in the group of size u obtain appropriate weights. The $u - 1$ "old" points obtain weights as a and the new point weights as a_0. The result is a 4-GDD of type 30^um^1 for $u \ge 5$, $u \notin M_6$ and $m \equiv 0 \pmod 3$ with $0 \le m \le 12(u-1) + 3(u-1) = 15(u-1)$. The possible exceptions for a_0 are not on the upper limit. Therefore, they can be compensated. □

Lemma 4.2. *There exists a* 4-GDD *of type* $30^u m^1$ *for* $u \in M_6$ *and* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 12(u-1)+9$. *There exists a* 4-GDD *of type* $30^u m^1$ *for* $u \in M_6$ *and* $m \equiv 0 \pmod{15}$ *with* $0 \le m \le 15(u-1)$.

Proof. There exists a 4-GDD of type $6^u m^1$, $u \in M_6$ and $m \equiv 0 \pmod 3$ with $0 \le m \le 3(u-1)$ by Theorem 1.5 without exceptions for $u \in M_6$, which is our master design.

We give each point weight 5, apply a 4-GDD of type 5^4 (Theorem 1.1) and obtain a 4-GDD of type $30^u(5m)^1$, which is the second assertion.

There exists a 4-DGDD$(6u, 6^u)^5$ for $u \in M_6$ by Theorem 1.8. Therefore with the master design, there exists a 4-GDD of type $30^u m^1$, $u \in M_6$ and $m \equiv 0 \pmod 3$ with $0 \le m \le 3(u-1)$ by Construction 1.9.

There exists a TD$(6, u+1)$, $u \in M_6$ by Theorem 1.6, and therefore there exists a $\{6, u+2\}$-GDD of type $5^{u+1}(u+1)^1$ by Construction 1.14. Removing a group of size 5, we obtain a $\{5, 6, u+1, u+2\}$-GDD of type $5^u(u+1)^1$. There exist 4-GDDs of types 6^5, 6^4a^1, 6^6, 6^5a^1, $a \equiv 0 \pmod 3$, $3 \le a \le 9$, and 4-GDDs of types $6^u a_0^1$, $6^{u+1}a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \le a_0 \le 3(u-1)$ by Theorem 1.5. The points in the group of size $u+1$ obtain appropriate weights. The u "old" points obtain weights as a and the new point weights as a_0. The result is a 4-GDD of type $30^u m^1$, $u \in M_6$ and $m \equiv 0 \pmod 3$ with $3u \le m \le 9u + 3(u-1) = 12(u-1)+9$. □

Lemma 4.3. *There exists a* 4-GDD *of type* $30^6 m^1$ *for* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 75$.

Proof. There exist a 4-GDD of type $30^6 m^1$ for $m \equiv 0 \pmod 3$ with $0 \le m \le 69$ by Lemma 4.2 and a 4-GDD of type $30^6 75^1$ by Theorem 1.2. Therefore, we need only a 4-GDD of type $30^6 72^1$. A $\{3, 4\}$-LGDD$_{15}$ of type 2^6, $\mathbf{G} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{11, 12\}\}$, with all blocks of size 3 in 72 3-pcs can be found in the online resource [36]. It results in a $\{3, 4\}$-GDD of type 30^6 with all blocks of size 3 in 72 3-pc by Theorem 1.15. Completing all 3-pcs we obtain the desired design. □

Lemma 4.4. *There exists a* 4-GDD *of type* $30^{18} m^1$ *for* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 15(18-1) = 255$.

Proof. There exists a 4-GDD of type $30^{18} m^1$ for $m \equiv 0 \pmod 3$ with $0 \le m \le 12\,(18-1)+9 = 205$ by Lemma 4.2.

There exist a 4-GDD of type 180^4 by Theorem 1.1 (the master design) and a 4-GDD of type $30^6 m_0^1$ for $m_0 \equiv 0 \pmod 3$, $0 \le m_0 \le 75$ (the ingredient design) by Lemma 4.3. Adjoin m_0 infinite points to the last group of the master design and fill all other groups of the master design with the ingredient design, where the infinite points form the group of size m_0. The result is a 4-GDD of type $30^{18} m^1$, $m \equiv 0 \pmod 3$, $180 \le m \le 255$. □

Lemma 4.5. *There exists a* 4-GDD *of type* $30^{22} m^1$ *for* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 14\,(22-1) = 294$.

Proof. There exists a 4-RGDD of type 2^{22} by Theorem 1.7. Completing the parallel classes results in a 5-GDD of type $2^{22}14^1$ our master design. There exists a 4-GDD of type $15^4 a^1$, $a \equiv 0 \pmod 3$, $0 \le a \le 21$ by Theorem 1.5. In the last group of the master design the points obtain appropriate weights. All other points weight 15. The result is a 4-GDD of type $30^{22} m^1$, $m \equiv 0 \pmod 3$ and $0 \le m \le 14 \cdot 21 = 294$. □

All lemmas of this section give:

Theorem 4.6. *There exists a* 4-GDD *of type* $30^u m^1$ *if, and only if,* $u \ge 4$ *and* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 15\,(u-1)$, *possibly excepting*

$u = 10$, $m \in \{123, 126, 129, 132\}$;
$u = 14$, $m \in \{168, 171, 174, 177, 183, 186, 189, 192\}$;
$u = 22$, $m \in \{297, 303, 306, 309, 312\}$.

5. Above 4-GDDs of type $18^u m^1$

In this section we develop several miscellaneous results, which we apply in the last sections.

Lemma 5.1 (*[17]*). *There exists a* 4-GDD *of type* $18^u m^1$ *for* $u \equiv 0 \pmod 4$, $u = 4$ *or* $u \ge 12$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 9\,(u-1)$, *except possibly when* $u = 12$ *and* $0 < m < 18$.

We have the following improvement for group size 18:

Lemma 5.2. *There exists a* 4-GDD *of type* $18^u m^1$ *for* $u \equiv 0 \pmod 4$, $u \ge 4$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 9\,(u-1)$, *except possibly when* $u = 8$ *and* $m \in \{12, 15\}$.

Proof. By reason of Lemma 5.1 we have only to look for the cases $u = 8$ and $u = 12$. There exists a 4-GDD of type $3^u m^1$ for $u \in \{8, 12\}$ and $m \equiv 0 \pmod 3$, $0 \le m \le (3(u-1)-3)/2$ by Theorem 1.5. Therefore, there exists a 4-GDD of type $18^u m^1$ for $u \in \{8, 12\}$ and $m \equiv 0 \pmod 3$, $0 \le m \le (3(u-1)-3)/2$ by Theorem 1.8 and Construction 1.9.

There exists a 4-RGDD of type 3^u for $u \in \{8, 12\}$. Completing the parallel classes results in a 5-GDD of type $3^u(u-1)^1$, which we use as our master design. We give all points of the last group weights 3, 6 or 9 and all other points weight 6. Filling in 4-GDDs of types $6^4 3^1$, $6^4 6^1$, $6^4 9^1$, which are given in Theorem 1.5, we obtain a 4-GDD of type $18^u m^1$ for $u \in \{8, 12\}$ and $m \equiv 0 \pmod 3$, $3(u-1) \le m \le 9(u-1)$. There exists a 4-GDD of type $18^9 \equiv 18^8\, 18^1$. Together we have a 4-GDD of type $18^8 m^1$, $m \in \{0, 3, 6, 9, 18, 21, \dots, 63\}$ and a 4-GDD of type $18^{12} m^1$, $m \equiv 0 \pmod 3$, $0 \le m \le 15$, and $18 \le m \le 99$ by Lemma 5.1. □

Lemma 5.3. *There exist a* 4-GDD *of type* $18^6\, 21^1$ *and a* 4-GDD *of type* $18^u m^1$ *for* $u \ge 5$, $m \equiv 0 \pmod 9$, $0 \le m \le 9(u-1)$ *possibly excepting* $(u, m) \in \{(11, 81), (13, 99), (19, 153)\}$.

Proof. A 4-GDD of type $18^6\, 21^1$ is given in [33].

There exists a 4-GDD of type $6^u m_0^1$ for $u \ge 5$ and $m_0 \equiv 0 \pmod 3$, $0 \le m_0 \le 3(u-1)$, *possibly excepting* $(u, m_0) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$ by Theorem 1.5. By Wilson's Fundamental Construction (WFC) we obtain a 4-GDD of type $18^u(3m_0)^1$ for $u \ge 5$ and $m_0 \equiv 0 \pmod 3$, $0 \le 3m_0 \le 9(u-1)$, possibly excepting the above values.

There exists a TD(4, u) for $u \ge 4$, $u \ne 6$ by Theorem 1.6. Remove a point and use this point to redefine the groups. Complete the groups of size u with a new point. This gives a $\{4, u+1\}$-GDD of type $3^u u^1$ as our master design. There exists a 4-GDD of type $6^u a_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \le a_0 \le 3\,(u-1)$ by Theorem 1.5 with some exceptions. We give every point in a group of size 3 in the master design the weight 6. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain 6 as weight and the new point weights as a_0. The result is a 4-GDD of type $18^{\,u} m^1$, $u \ge 4$, $u \ne 6$, $m \equiv 0 \pmod 3$ and $6\,(u-1) \le m \le 6\,(u-1) + 3\,(u-1) = 9\,(u-1)$ but with some exceptions. The result is a 4-GDD of type $18^{\,u} m^1$ for $(u, m) \in \{(7, 45), (11, 63), (11, 72), (13, 81), (17, 117), (17, 126), (19, 135), (19, 144), (23, 180), (23, 189)\}$. □

Lemma 5.4. *There exist* 4-GDD *s of types* $18^5 6^1$, $18^5\, 24^1$, $18^5\, 30^1$.

Proof. There exists a 4-GDD of type $3^5 6^1$ by Theorem 1.5. Therefore, there exists a 4-GDD of type $18^5 6^1$ by Theorem 1.8 and Construction 1.9.

We give a $\{3, 4\}$-LGDD$_6$ of type 3^5 with all blocks of size 3 in $m \in \{24, 30\}$ 3-pcs (each 3-pc is a row), $\mathbf{G} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}, \{13, 14, 15\}\}$ in [36]. This results in a $\{3, 4\}$-GDD of type 18^5, with all blocks of size 3 in $m \in \{24, 30\}$ 3-pcs by Theorem 1.15. Completing all 3-pcs we obtain 4-GDDs of types $18^5 24^1$ and $18^5 30^1$. □

Lemma 5.5. *There exists a* 4-GDD *of type* $10^u m^1$ *for each* $u \ge 12$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ *and* $10 \le m \le 5(u-1)$, *possibly excepting*

$$u = 30, \quad m \in \{133, 136, 139, 142\};$$
$$u = 42, \quad m \in \{178, 181, 184, 187, 193, 196, 199, 202\};$$
$$u = 66, \quad m \in \{307, 313, 316, 319, 322\}.$$

Proof. There exists a 4-GDD of type $30^u m^1$ for $u \ge 4$, $m \equiv 0 \pmod 3$ and $0 \le m \le 24(u-1)$, $0 \le m \le 15(u-1)$, possibly excepting $u = 10$, $m \in \{123, 126, 129, 132\}$;
$u = 14$, $m \in \{168, 171, 174, 177, 183, 186, 189, 192\}$; $u = 22$,
$m \in \{297, 303, 306, 309, 312\}$ by Theorem 4.6. Adjoin 10 infinite points and fill all groups of size 30 with a 4-GDD of type 10^4 (Theorem 1.1), where the infinite points form a group. This gives a 4-GDD of type $10^{3u}(m+10)^1$ for $u \ge 4$, $m \equiv 0 \pmod 3$ and $10 \le m + 10 \le 15(u-1) + 10 = 5(3u-1)$ with above exceptions. □

6. Above 4-GDDs of type $90^u m^1$

Lemma 6.1. *There exists a* 4-GDD *of type* $90^u 3^1$ *for* $u \ge 4$.

Proof. There exists a 4-GDD of type $6^u 3^1$ for $u \ge 4$ by Theorem 1.5 and therefore, a 4-GDD of type $90^u 3^1$ *for* $u \ge 4$ by Theorem 1.8 and Construction 1.9. □

Lemma 6.2. *There exists a* 4-GDD *of type* $90^6 m^1$ *for* $m \equiv 0 \pmod 3$ *with* $0 \le m \le 45\,(u-1)$, *except possibly when* $m \in \{210, 213, 219, 222\}$.

Proof. The case $m = 3$ is shown in Lemma 6.1. There exists a TD(6, 7) by Theorem 1.6 and we obtain a $\{6, 8\}$-GDD of type $5^7\,7^1$ by Construction 1.14. Deleting all points from one group of size 5 we get a $\{5, 6, 7, 8\}$-GDD of type $5^6\,7^1$ as our master design. There exist 4-GDDs of types 18^4a^1, 18^5a^1, $a \in \{0, 6, 9, 18, 24, 27\}$, $18^6a_0^1$, $18^7a_0^1$, $a_0 \equiv 0 \pmod 9$, $0 \le a_0 \le 9(6-1) = 45$ by Lemmas 5.3 and 5.4. We give every point in a group of size 5 in the master design weight 18. The points in the group of size 7 obtain appropriate weights. The result is a 4-GDD of type 90^6m^1, $m \equiv 0 \pmod 3$ and

$$0 \le m \le \begin{cases} 27 \cdot 6 + 45 = 207 = 41 \cdot 5 + 2 & \text{for } m \equiv 0 \pmod 9 \\ 27 \cdot 5 + 24 + 45 = 204 = 40 \cdot 5 + 4 & \text{for } m \equiv 6 \pmod 9 \\ 27 \cdot 4 + 24 \cdot 2 + 45 = 201 = 40 \cdot 5 + 1 & \text{for } m \equiv 3 \pmod 9). \end{cases}$$

There exist 4-GDDs of types 30^672^1, 30^675^1 by Theorem 4.6 and therefore 4-GDDs of types 90^6216^1, 90^6225^1 by WFC. □

Let again $M_6 = \{6, 10, 14, 18, 22\}$. Then there exists a TD$(6, u)$ for $u \ge 5$ with $u \notin M_6$ by Theorem 1.6.

Lemma 6.3. *There exist a* 4-GDD *of type* 90^um^1 *for* $u \ge 5$, $u \notin M_6$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 45\,(u-1) - 12$ *and a* 4-GDD *of type* 90^um^1 *for* $u \ge 5$, $u \in M_6 \setminus 6$, $m \equiv 0 \pmod 3$ *with* $0 \le m \le 42\,(u-1)$.

Proof. The case $m = 3$ is shown in Lemma 6.1. For $u \ge 5$, $u \notin M_6$ there exists a TD$(6, u)$ and therefore a $\{6, u+1\}$-GDD of type 5^uu^1 by Construction 1.14 as our master design. There exist a 4-GDD of type 18^5a^1, $a \in \{0, 6, 9, 18, 24, 27, 30, 36\}$ and a 4-GDD of type $18^ua_0^1$, $a_0 \equiv 0 \pmod 9$, $0 \le a_0 \le 9(u-1)$ with some possible exceptions by Lemmas 5.3 and 5.4. We give every point in a group of size 5 in the master design weight 18. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain weights as a and the new point weights as a_0. The result is a 4-GDD of type 90^um^1 for $u \ge 5$, $u \notin M_6$, $m \equiv 0 \pmod 3$ with

$$0 \le m \le \begin{cases} 36(u-1) + 9(u-1) = 45(u-1) & \text{for } m \equiv 0 \pmod 9 \\ 36(u-2) + 30 + 9(u-1) = 45(u-1) - 6 & \text{for } m \equiv 6 \pmod 9 \\ 36(u-3) + 30 \cdot 2 + 9(u-1) = 45(u-1) - 12 & \text{for } m \equiv 3 \pmod 9). \end{cases}$$

The possible exceptions for a_0 are not on the lower or upper limit. Therefore, it is not necessary to apply these values.

For $u \in M_6 \setminus 6$ there exists a 4-RGDD of type 6^u by Theorem 1.7. Completing results in a 5-GDD of type $6^u(2(u-1))^1$ our master design. There exists a 4-GDD of type 15^4a^1, $a \equiv 0 \pmod 3$, $0 \le a \le 21$ by Theorem 1.5. In the last group of the master design the points obtain appropriate weights. All other points weight 15. The result is a 4-GDD of type 90^um^1, $m \equiv 0 \pmod 3$ and $0 \le m \le 2(u-1) \cdot 21 = 42(u-1)$. □

Lemma 6.4. *There exists a* 4-GDD *of type* 90^um^1 *for* $u \ge 5$, $u \ne 6$, $m \equiv 0 \pmod 3$ *with* $30(u-1) \le m \le 45\,(u-1)$, *possibly excepting*

$u = 10$, $m \in \{123, 126, 129, 132\} + 30(u-1)$;
$u = 14$, $m \in \{168, 171, 174, 177, 183, 186, 189, 192\} + 30(u-1)$;
$u = 22$, $m \in \{297, 303, 306, 309, 312\} + 30(u-1)$.

Proof. For $u \ge 5$, $u \ne 6$ there exists a TD$(4, u)$ and therefore a $\{4, u+1\}$-GDD of type 3^uu^1 by Construction 1.14 as our master design. There exist a 4-GDD of type 30^4, and a 4-GDD of type $30^ua_0^1$, $a_0 \equiv 0 \pmod 3$, $0 \le a_0 \le 15(u-1)$ by Theorem 4.6 with some possible exceptions for a_0. We give every point in a group of size 3 in the master design weight 30. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain 30 as weight and the new point weights as a_0. The result is a 4-GDD of type 90^um^1 for $u \ge 5$, $u \ne 6$, $m \equiv 0 \pmod 3$ with $30(u-1) \le m = 30(u-1) + a_0 \le 30(u-1) + 15(u-1) = 45\,(u-1)$, possibly excepting

$u = 10$, $a_0 \in \{123, 126, 129, 132\}$;
$u = 14$, $a_0 \in \{168, 171, 174, 177, 183, 186, 189, 192\}$;
$u = 22$, $a_0 \in \{297, 303, 306, 309, 312\}$. □

Theorem 1.3 and all lemmas of this section result in:

Theorem 6.5. *A* 4-GDD *of type* 90^um^1 *exists if, and only if,* $u \ge 4$, $m \equiv 0 \pmod 3$ *and* $0 \le m \le 45\,(u-1)$, *possibly excepting*

$u = 6$, $m \in \{210, 213, 219, 222\}$;
$u = 10$, $m \in \{393, 396, 399, 402\}$;
$u = 14$, $m \in \{558, 561, 564, 567, 573, 576, 579, 582\}$;
$u = 22$, $m \in \{927, 933, 936, 939, 942\}$.

7. Above 4-GDDs of types $40^u m^1$ and $120^u m^1$

Theorem 7.1. *A* 4-GDD *of type* $120^u m^1$ *exists if, and only if,* $u \geq 4$, $m \equiv 0 \pmod 3$ *and* $0 \leq m \leq 60\,(u-1)$.

Proof. There exists a TD$(5, u)$ for $u \geq 4$ and $u \notin \{6, 10\}$ by Theorem 1.6. Remove a point and use this point to redefine the groups. Complete all groups of size u with a new point. This gives a $\{5, u+1\}$-GDD of type $4^u u^1$ our master design.

By Theorem 4.6 there exist 4-GDDs of types 30^5, $30^4 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 45$, and a 4-GDD of type $30^u a_0^1$ if, and only if, $u \geq 4$ and $a_0 \equiv 0 \pmod 3$ with $0 \leq a_0 \leq 15(u-1)$ and some exceptions. We give every point in a group of size 4 in the master design weight 30. The points in the group of size u obtain appropriate weights. The $u-1$ "old" points obtain weights as a and the new point as a_0. The result is a 4-GDD of type $120^u m^1$ for $u \geq 4$, $u \notin \{6, 10\}$ and $m \equiv 0 \pmod 3$, $0 \leq m \leq 45(u-1) + 15(u-1) = 60(u-1)$. The possible exceptions for a_0 are not on the lower or upper limit. Therefore, it is not necessary to apply these values.

There exists a 4-HTD of hole type $10^6 a^1$, $0 \leq a \leq 25$ by Theorem 1.11 and, therefore, a $\{3, 4\}$-DGDD of type $(180, 30^6)^4$ whose blocks of size 3 can be partitioned into $9a$ parallel classes by Construction 1.12. Adjoin $9a$ infinite points to complete the 3-pcs and then adjoin a further m ideal points, filling in 4-GDDs of type $30^6 m^1$, $m \equiv 0 \pmod 3$, $0 \leq m \leq 75$ coming from Theorem 4.6 to obtain a 4-GDD of type $120^6 (9a+m)^1$, $0 \leq a \leq 25$, $0 \leq m \leq 75$.

By Theorem 1.7 there exists a 4-RGDDof type 4^{10}. Completing all parallel classes results in a 5-GDD of type $4^{10} 12^1$ which we take as our master design. There exist 4-GDDs of types 30^5, $30^4 a^1$, $a \equiv 0 \pmod 3$, $0 \leq a \leq 45$ by Theorem 4.6. We give every point in a group of size 4 in the master design weight 30. The points in the group of size 12 obtain appropriate weights. The result is a 4-GDD of type $120^{10} m^1$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 12 \cdot 45 = 60 \cdot 9$. □

Theorem 7.2. *There exists a* 4-GDD *of type* $40^u m^1$ *for each* $u \geq 12$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ *and* $1 \leq m \leq 20(u-1)$.

Proof. There exists a 4-GDD of type $120^{\hat{u}} m^1$ for $\hat{u} \geq 4$, $m \equiv 0 \pmod 3$ and $0 \leq m \leq 60(\hat{u}-1)$ by Theorem 7.1. Adjoin 40 infinite points and fill all groups of size 120 with a 4-GDD of type 40^4 (Theorem 1.1), where the infinite points form a group. This gives a 4-GDD of type $40^{3\hat{u}}(m+40)^1$ for $\hat{u} \geq 4$, $m \equiv 0 \pmod 3$ and $40 \leq m + 40 \leq 60(\hat{u}-1) + 40 = 20(3\hat{u}-1)$.

There exists a 4-GDD of type $10^u m^1$ for each $u \geq 12$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ and $10 \leq m \leq 40$ by Lemma 5.5. Therefore, there exists a 4-GDD of type $40^u m^1$ for each $u \geq 12$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ and $10 \leq m \leq 40$ by Corollary 1.10.

There exists a 4-GDD of type $4^u m^1$ for each $u \geq 6$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ and $1 \leq m \leq 2(u-1)$ by Theorem 1.5. Therefore, there exists with $n = 10$ a 4-GDD of type $40^u m^1$ for each $u \geq 6$, $u \equiv 0 \pmod 3$, $m \equiv 1 \pmod 3$ and $1 \leq m \leq 2(u-1)$ by Theorem 1.8 and Construction 1.9. □

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