Small Uniformly Resolvable Designs for Block Sizes 3 and 4

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Abstract: A uniformly resolvable design (URD) is a resolvable design in which each parallel class contains blocks of only one block size k, such a class is denoted k-pc and for a given k the number of k-pcs is denoted r_k. In this paper, we consider the case of block sizes 3 and 4 (both existent). We use v to denote the number of points, in this case the necessary conditions imply that $v \equiv 0 \pmod{12}$. We prove that all admissible URDs with $v < 200$ points exist, with the possible exceptions of 13 values of r_4 over all permissible v. We obtain a URD({3, 4}; 276) with $r_4 = 9$ by direct construction use it to and complete the construction of all URD({3, 4}; v) with $r_4 = 9$. We prove that all admissible URDs for $v \equiv 36 \pmod{144}$, $v \equiv 0 \pmod{60}$, $v \equiv 36 \pmod{108}$, and $v \equiv 24 \pmod{48}$ exist, with a few possible exceptions. Recently, the existence of URDs for all admissible parameter sets with $v \equiv 0 \pmod{48}$ was settled, this together with the latter result gives the existence all admissible URDs for $v \equiv 0 \pmod{24}$, with a few possible exceptions. © 2013 Wiley Periodicals, Inc. J. Combin. Designs 21: 481–523, 2013

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1. INTRODUCTION

Let v and λ be positive integers, let K and M be two sets of positive integers. A *group divisible design*, denoted by $\mathrm{GDD}_\lambda(K, M; v)$, is a triple $(X, \boldsymbol{G}, \boldsymbol{B})$, where X is a set with v elements (called points), $\boldsymbol{G}$ is a set of subsets (called groups) of X, $\boldsymbol{G}$ partitions X, and $\boldsymbol{B}$ is a set of subsets (called blocks) of X such that

1. $|B| \in K$ for each $B \in \boldsymbol{B}$,
2. $|G| \in M$ for each $G \in \boldsymbol{G}$,
3. $|B \cap G| \leq 1$ for each $B \in \boldsymbol{B}$ and each $G \in \boldsymbol{G}$,
4. Each pair of elements of X from distinct groups is contained in exactly λ blocks.

The notation is similar to [3,4]. If $\lambda = 1$, the index λ is omitted. If $K = \{k\}$, respectively, $M = \{m\}$, then the $\mathrm{GDD}_\lambda(K, M; v)$ is simply denoted by $\mathrm{GDD}_\lambda(k, M; v)$, respectively, $\mathrm{GDD}_\lambda(K, m; v)$, which may also be specified in "exponential" form as *K-GDD*$_\lambda$ of type $m^{v/m}$. A $\mathrm{GDD}_\lambda(K, 1; v)$ is called a *pairwise balanced design* and denoted by $\mathrm{PBD}_\lambda(K; v)$.

Theorem 1.1 ([17,22]). *There exists a* 4-*GDDof type* g^4m^1 *with* $m > 0$ *if and only if* $g \equiv m \equiv 0 \pmod 3$ *and* $0 < m \leq 3g/2$.

Theorem 1.2 ([1, 14, 22]). *There exists a* 5-*GDD of type* g^5m^1 *with* $m > 0$ *if* $g \equiv m \equiv 0 \pmod 4$ *and* $0 < m \leq 4g/3$*, with the possible exceptions of* $(g, m) = (12, 4)$ *and* $(12, 8)$.

A *transversal design* $\mathrm{TD}_\lambda(k, g)$ is equivalent to a $\mathrm{GDD}_\lambda(k, g; kg)$. That means, in a $\mathrm{TD}_\lambda(k, g)$, each block contains a point from each group. If $\lambda = 1$, the index λ is omitted.

Theorem 1.3 ([2]). *A* $\mathrm{TD}(k, g)$ *exists in the following cases:*

1. $k = 6$ *and* $g \geq 5$ *and* $g \notin \{6, 10, 14, 18, 22\}$;
2. $k = 7$ *and* $g \geq 7$ *and* $g \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$;
3. *A* $\mathrm{TD}(p + 1, p)$ *exists, where* p *is a prime power.*

In a $\mathrm{GDD}_\lambda(K, M; v)$ with $(X, \boldsymbol{G}, \boldsymbol{B})$, a *parallel class* is a set of blocks, which partitions X. If $\boldsymbol{B}$ can be partitioned into parallel classes, then the $\mathrm{GDD}_\lambda(K, M; v)$ is said to be *resolvable* and denoted by $\mathrm{RGDD}_\lambda(K, M; v)$. Analogously, a resolvable $\mathrm{PBD}_\lambda(K; v)$ is denoted by $\mathrm{RPBD}_\lambda(K; v)$. A parallel class is said to be *uniform* if it contains blocks of only one size k (k-pc). If all parallel classes of an $\mathrm{RPBD}_\lambda(K; v)$ are uniform, the design is said to be *uniformly resolvable*. Here, a uniformly resolvable design $\mathrm{RPBD}_\lambda(K; v)$ is denoted by $\mathrm{URD}_\lambda(K; v)$. If $\lambda = 1$, the index λ is omitted. In a $\mathrm{URD}_\lambda(K; v)$, the number of resolution classes with blocks of size k is denoted r_k, $k \in K$. Uniformly resolvable designs with block sizes 3 and 4 mean here $\mathrm{URD}(\{3, 4\}; v)$ with $r_3 > 0$ and $r_4 > 0$.

The following theorem about RGDDs will be applied later.

Theorem 1.4 ([4, 9–13, 16, 18, 23, 27, 29, 31, 32]). *The necessary conditions for the existence of a k-RGDD of type* h^n, $\mathrm{RGDD}(k, h; hn)$*, namely,* $n \geq k$, $hn \equiv 0 \pmod k$*, and* $h(n - 1) \equiv 0 \pmod{k - 1}$*, are also sufficient for*

$k = 2$;
$k = 3$*, except for* $(h, n) \in \{(2, 3), (2, 6), (6, 3)\}$*; and for*
$k = 4$*, except for* $(h, n) \in \{(2, 4), (2, 10), (3, 4), (6, 4)\}$ *and possibly excepting:*

1. $h \equiv 2, 10 \pmod{12}$:
   $h = 2$ *and* $n \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 178, 184, 202, 214, 238, 250, 334\}$;
   $h = 10$ *and* $n \in \{4, 34, 52, 94\}$;
   $h \in [14, 454] \cup \{478, 502, 514, 526, 614, 626, 686\}$ *and* $n \in \{10, 70, 82\}$.
2. $h \equiv 6 \pmod{12}$: $h = 6$ *and* $n \in \{6, 68\}$; $h = 18$ *and* $n \in \{18, 38, 62\}$.
3. $h \equiv 9 \pmod{12}$: $h = 9$ *and* $n = 44$.
4. $h \equiv 0 \pmod{12}$: $h = 24$ *and* $n = 23$; $h = 36$ *and* $n \in \{11, 14, 15, 18, 23\}$.

A *resolvable transversal design* $\mathrm{RTD}_\lambda(k, g)$ is equivalent to an $\mathrm{RGDD}_\lambda(k, g; kg)$. That means, each block in an $\mathrm{RTD}_\lambda(k, g)$ contains a point from each group. A *K-frame* is a GDD $(X, \boldsymbol{G}, \boldsymbol{B})$ with index unity, in which the collection of blocks $\boldsymbol{B}$ can be partitioned into holey parallel classes each of which partitions $X \setminus G$ for some $G \in \boldsymbol{G}$. We use the usual exponential notation for the types of GDDs and frames. Thus, a GDD or a frame of type $1^i 2^j \ldots$ is one in which there are i groups of size 1, j groups of size 2, and so on. A K-frame is called *uniform* if each partial parallel class is of only one block size. It is called *completely uniform* if for each hole G the resolution classes which partition $X \setminus G$ are all of one block size. We use mostly $K = \{3, 4\}$. A $\{3, 4\}$-frame of type $(g; 3^{n_1} 4^{n_2})^u (m; 3^{n_3} 4^{n_4})^1$ has u groups of size g. Each group of size g has n_1 holey pcs of block size 3 and n_2 holey pcs of block size 4. The only group of size m has n_3 holey pcs of block size 3 and n_4 holey pcs of block size 4.

Theorem 1.5 ([23]). *For $k = 2$ and $k = 3$, there exists a k-frame of type h^u if and only if $u \geq k + 1$, $h \equiv 0 \pmod{k-1}$, and $h \cdot (u - 1) \equiv 0 \,(\mathrm{mod}\, k)$.*

Theorem 1.6 ([8,13,15,16,19,23,33]). *There exists a 4-frame of type h^u if and only if $u \geq 5$, $h \equiv 0 \pmod 3$ and $h\,(u - 1) \equiv 0 \pmod 4$, except possibly where*

1. *$h = 36$ and $u = 12$;*
2. *$h \equiv 6(\mathrm{mod}\,12)$:*
   $h = 6$ and $u \in \{7, 23, 27, 35, 39, 47\}$;
   $h = 18$ and $u \in \{15, 23, 27\}$;
   $h \in \{30\} \cup [66,\ 2, 190]$ and $u \in \{7, 23, 27, 39, 47\}$;
   $h \in \{42,\ 54\} \cup$ [2,202, 11,238] and $u \in \{23, 27\}$.

We will also use incomplete group divisible designs (IGDDs). An IGDD with block sizes from a set K and index unity is a quadruple (X, $\boldsymbol{G}$, H, $\boldsymbol{B}$), which meets the following conditions:

1. $\boldsymbol{G} = \{G_1, G_2, \ldots, G_n\}$ is a partition of the set X of points into subsets called *groups*,
2. H is a subset of X called the *hole*,
3. $\boldsymbol{B}$ is a collection of subsets of X with cardinalities from K, called *blocks*, so that a group and a block contain at most one common point,
4. every pair of points from distinct groups is either in H or occurs in a unique block but not both.

This design is denoted by IGDD$(K, M; v)$ of type T, where $M = \{|G_1|, |G_2|, \ldots, |G_n|\}$ and T is the multiset $\{(|G_i|, |G_i \cap H|) : 1 \leq i \leq n\}$. Sometimes "exponential" notation is used to describe the type. An IGDD$(K, M; v)$ of type T is said to be *uniformly resolvable* and denoted by IUGDD$(K, M; v)$ of type T if blocks can be partitioned into uniform parallel classes and partial uniform parallel classes, the latter partitioning $X \setminus H$. The numbers of uniform parallel classes, partial uniform parallel classes with blocks of size k are denoted by r_k, r_k°, respectively. If $|G_i| = 1$ for $1 \leq i \leq n$, then the IUGDD is denoted *incomplete uniformly resolvable design* IURD$(K; v)$ *with a hole H*.

Some known results about URDs are summarized below. Rees [20] introduced URDs and showed:

Theorem 1.7 ([20]). *There exists a* URD$(\{2, 3\}; v)$ *with $r_2, r_3 > 0$ if and only if*

1. *$v \equiv 0\,(\mathrm{mod}\,6)$;*
2. *$r_2 = v - 1 - 2r_3 (r_3 = \frac{v-1-r_2}{2})$;*
3. *$1 \leq r_3 \leq \frac{v}{2} - 1$;*

with the two exceptions $(v, r_3) = (6, 2), (12, 5)$.

Recently, almost all URDs with $K = \{2, 4\}$ were constructed in [7] and slightly improved in [27] as follows:

Theorem 1.8. *There exists a URD*$(\{2, 4\};\ v)$ *with* $r_2, r_4 > 0$ *if and only if*

1. $v \equiv 0\,(\mathrm{mod}\,4)$*;*
2. $r_2 = v - 1 - 3r_4$ $(r_4 = \frac{v-1-r_2}{3})$*;*

with two exceptions $(v, r_2) = (8, 1), (20, 1)$ *and possibly excepting:*
$(v, r_2) = (2n, 1)$*,* $n \in \{52, 100, 184\}$*;*
$(v, r_2) = (2n, r_2)$*,* $n \in \{34, 46, 70, 82, 94, 118, 130, 178, 202, 214, 238, 250, 334\}$*,* r_2 *admissible;*
$(v, r_2) = (12n, 2)$*,* $n \in \{2, 7, 9, 10, 11, 13, 14, 17, 19, 22, 31, 34, 38, 43, 46, 47, 82\}$.

Theorem 1.9 ([6]). *The necessary conditions for the existence of a URD*$(\{3,\ 4\};\ v)$ *with* $r_3, r_4 > 0$ *are*

- $v \equiv 0\,(\mathrm{mod}\,12)$*;*
- r_4 *is odd;*
- *if* $r_k > 1$*, then* $v \geq k^2$*; and*
- $r_4 = \frac{v-1-2r_3}{3}$ $(r_3 = \frac{v-1-3r_4}{2})$.

The fourth condition means that if r_3 is given, then r_4 is determined, and vice versa. It also implies that $r_3 \leq (v/2) - 2$ and $r_4 \leq (v/3) - 1$.

Remark. $r_3 \equiv 1(\mathrm{mod}\,3)$.

Proof. Because r_4 is odd, insert $2i + 1$ for r_4 in the last equation of Theorem 1.9; this gives $r_3 = \frac{v}{2} - 3i - 2 \equiv -2 \equiv 1(\mathrm{mod}\,3)$. □

We will now summarize some known results of URDs with block sizes 3 and 4. The next two theorems are special cases of Theorem 1.4. We take the groups as an additional parallel class to get the URDs.

Theorem 1.10 ([25]). *There exist an RGDD*$(3, 4;\ v)$ *and equivalently a URD*$(\{3,\ 4\};\ v)$ *with* $r_4 = 1$ *if and only if* $v \equiv 0\,(\mathrm{mod}\,12)$.

Theorem 1.11 ([21, 23, 29, 31]). *There exist an RGDD*$(4, 3;\ v)$ *and equivalently a URD*$(\{3,\ 4\};\ v)$ *with* $r_3 = 1$ *if and only if* $v \equiv 0\,(\mathrm{mod}\,12)$*,* $v \geq 24$.

Theorem 1.12 ([5, 24, 27]). *There exists a URD*$(\{3,\ 4\};\ v)$ *with* $r_4 = 3, 5,$ *or* 7 *if and only if* $v \equiv 0\,(\mathrm{mod}\,12)$*, except when* $v = 12$*. There exists a URD*$(\{3, 4\};\ v)$ *with* $r_4 = 9$ *if and only if* $v \equiv 0\,(\mathrm{mod}\,12)$ *except* $v = 12, 24$ *and except possibly when* $v = 276$.

There exist also results for small r_3.

Theorem 1.13 ([27]). *There exists a URD*$(\{3,\ 4\};\ v)$ *with* $r_3 = 4$ *if and only if* $v \equiv 0\,(\mathrm{mod}\,12)$*. There exists a URD*$(\{3,\ 4\};\ v)$ *with* $r_3 = 7$ *if and only if* $v \equiv 0\,(\mathrm{mod}\,12)$*, except when* $v = 12$*, and possibly excepting the following 11 values:* $v \in \{72, 84, 108, 132, 156, 204, 228, 276, 348, 372, 444\}$.

There exists a URD$(\{3, 4\};\ v)$ *with* $r_3 = 10$ *if and only if* $v \equiv 0\,(\mathrm{mod}\,12)$*, except when* $v = 12$*, and possibly excepting the following 12 values:* $v \in \{60, 72, 108, 132, 156, 204, 228, 276, 300, 348, 372, 492\}$.

The main result in [27] is as follows:

Theorem 1.14. *For* $v \equiv 0 \pmod{48}$*, all admissible URD*$(\{3,4\};\ v)$ *exist.*

Further, the following result will be applied later.

Lemma 1.15 ([27]). *There exists a uniformly resolvable*{3,4}-RGDD *of type* 12^4 *with* $r_4 \in \{0, 2, 4, 6, 8, 12\}$ *(and* $r_3 \in \{18, 15, 12, 9, 6, 0\}$*).*

There is also a result for $K = \{3, 5\}$.

Theorem 1.16 ([25–27]). *There exists a URD*$(\{3,5\}; v)$ *with* $r_5 = 2, 3, 4, 5$ *if and only if* $v \equiv 15 \pmod{30}$ *except* $v = 15$.

We use the concept of labeled resolvable designs to get direct constructions for resolvable designs. This concept was introduced by Shen [28,30,31].

Let $(X, \boldsymbol{B})$ be a (UR)GDD$_\lambda(K, M; v)$ where $X = \{a_1, a_2, \ldots, a_v\}$ is totally ordered with ordering $a_1 < a_2 < \cdots < a_v$. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, we suppose that $x_1 < x_2 < \cdots < x_k$. Let Z_λ be the group of residues modulo λ.

Let $\varphi : \boldsymbol{B} \to Z_\lambda^{\binom{k}{2}}$ be a mapping where for each $B = \{x_1, x_2, \ldots, x_k\} \in \boldsymbol{B}$, $k \in K$,

$$\varphi(B) = (\varphi(x_1, x_2), \ldots, \varphi(x_1, x_k), \varphi(x_2, x_3), \ldots, \varphi(x_2, x_k), \varphi(x_3, x_4), \ldots, \varphi(x_{k-1}, x_k)),$$
$$\varphi(x_i, x_j) \in Z_\lambda \quad \text{for} \quad 1 \le i < j \le k.$$

A (UR)GDD$_\lambda(K, M; v)$ is said to be a *labeled (uniform resolvable) group divisible design*, denoted by L(U)GDD$_\lambda(K, M; v)$, if there exists a mapping φ such that:

1. For each pair$\{x, y\} \subset X$ with $x < y$, contained in the blocks $B_1, B_2, \ldots, B_\lambda$, then $\varphi_i(x, y) \equiv \varphi_j(x, y)$ if and only if $i = j$ where the subscripts i and j denote the blocks to which the pair belongs, for $1 \le i, j \le \lambda$; and
2. For each block $B = \{x_1, x_2, \ldots, x_k\}$, $k \in K$, $\varphi(x_r, x_s) + \varphi(x_s, x_t) \equiv \varphi(x_r, x_t) \pmod{\lambda}$, for $1 \le r < s < t \le k$.

The blocks will be denoted in the following form:

$$(x_1\ x_2\ \ldots\ x_k; \varphi(x_1, x_2)\ \ldots \varphi(x_1, x_k)\ \varphi(x_2, x_3) \ldots\ \varphi(x_2, x_k)\ \varphi(x_3, x_4) \ldots\ \varphi(x_{k-1}, x_k)),$$
$$k \in K.$$

The above definition was first given in [24] and is a little bit more general than the definition by Shen [31] with $K = \{k\}$ or Shen and Wang [30] for transversal designs. A special case of type 1^v, a labeled URD$_\lambda(K; v)$, is denoted by LURD$_\lambda(K; v)$. A labeled K-frame of type T and index λ is denoted by $K-$LF$_\lambda$ of type T.

The main application of the labeled designs is to blow up the point set of a given design with the following theorem (Shen [16]) here extended for labeled (uniform resolvable) pairwise balanced designs.

Theorem 1.17 ([16,24]). *If there exists an L(U)GDD*$_\lambda(K, M;\ v)$ *(with* r_k^L *classes of size k, for each* $k \in K$*), then there exists a (U)GDD*$(K, \lambda M;\ \lambda v)$*, where* $\lambda M = \{\lambda g_i | g_i \in M\}$ *(with* $r_k = r_k^L$ *classes of size k, for each* $k \in K$*). If there exists a uniform frame K-LF*$_\lambda$ *of type T, then there exists a uniform K-frame of type* λT*, where* $\lambda T = \{\lambda g_i | g_i \in T\}$.

A special case for URDs is shown in the following.

Corollary 1.18. *If there exists an $LURD_\lambda(K;\ v)$ with r_k^L classes of size k, for each $k \in K$, then there exists a URD($K \cup \{\lambda\};\ \lambda\, v$) with $r_k = r_k^L$ when $k \neq \lambda$, and $r_\lambda = r_\lambda^L + 1$, where we take $r_\lambda^L = 0$ if $\lambda \notin K$.*

A K-uniform *semiframe* of type g^u and index λ is a K-GDD_λ of type g^u $(X, \boldsymbol{G}, \boldsymbol{B})$, in which the collection of blocks $\boldsymbol{B}$ can be written as a disjoint union $\boldsymbol{B} = \boldsymbol{P} \cup \boldsymbol{F}$, where $\boldsymbol{F}$ is partitioned into uniform parallel classes of X and $\boldsymbol{P}$ is partitioned into uniform partial parallel classes, where each uniform partial parallel class is a partition of X/G for some $G \in \boldsymbol{G}$. The number of partial classes per group in a frame or semiframe of size k will be indicated by a tilde, $\tilde{r}_k$. A semiframe is called *perfectly uniform* if there are two block sizes and P are all of one size and F are all of the other. A labeled (perfectly) uniform semiframe is a semiframe with a labeling on the blocks as above. It is worth noting that, in general, a frame or semiframe may have different numbers of classes of each size missing different groups, we exploit this fact in many of our constructions.

Analogously to Theorem 1.17, we obtain.

Theorem 1.19. *If there exists a labeled (perfectly) uniform semiframe K-LSF_λ of type T, then there exists a (perfectly) uniform K-semiframe of type λT, where $\lambda T = \{\lambda g_i | g_i \in T\}$.*

In Section 2, some small {3, 4}-URGDDs are directly constructed. All URDs with $v < 200$ point are examined in Section 3. Required {3, 4}-URGDDs and {3, 4}-frames are contained in Section 4. The most important results of Section 5 are that there exist all admissible URDs for $v \equiv 0 \pmod{60}$ for $v > 120$ and $v \equiv 36 \pmod{108}$. In Section 6, we consider the case where $v \equiv 24 \pmod{48}$. We show that all URDs with $v \equiv 24 \pmod{48}$ exist with a few possible exceptions.

2. DIRECT CONSTRUCTIONS

The following desired designs were found computationally.

Lemma 2.1. *There exists a uniformly resolvable $\{3, 4\} - URGDD$ of type 6^4 with $r_4 \in \{0, 2, 4\}$.*

Proof. There exists a 3-RGDD of type 6^4 by Theorem 1.4. Let

$$\boldsymbol{G} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}.$$

A uniformly resolvable $\{3,4\} - \mathrm{LRGDD}_2$ of type 3^4 with $r_3 = 6$ and $r_4 = 2$; each row forms a parallel class:

(6 7 12; 1 1 0), (2 8 11; 0 0 0), (1 5 9; 1 0 1), (3 4 10; 0 1 1),

(2 6 8; 1 1 0), (4 9 10; 0 0 0), (1 5 11; 0 1 1), (3 7 12; 1 0 1),

(1 6 11; 0 0 0), (2 4 7; 0 0 0), (3 9 12; 0 1 1), (5 8 10; 1 0 1),

(4 9 11; 1 1 0), (2 5 12; 1 1 0), (3 6 8; 0 1 1), (1 7 10; 1 0 1),

(5 8 12; 0 1 1), (1 4 7; 1 0 1), (3 6 10; 1 0 1), (2 9 11; 0 1 1),

(2 4 12; 1 0 1), (6 7 11; 0 1 1), (1 8 10; 1 1 0), (3 5 9; 1 1 0),

(2 5 7 10; 0 1 1 1 1 0), (1 6 9 12; 1 1 1 0 0 0), (3 4 8 11; 1 0 1 1 0 1),
(2 6 9 10; 0 1 0 1 0 1), (1 4 8 12; 0 0 0 0 0 0), (3 5 7 11; 0 0 0 0 0 0).

A uniformly resolvable {3,4} $-$ LRGDD$_2$ of type 3^4 with $r_3 = 3$ and $r_4 = 4$; each row forms a parallel class:

(1 7 10; 0 1 1), (3 6 9; 1 1 0), (2 5 12; 1 1 0), (4 8 11; 0 0 0),
(5 9 10; 0 0 0), (2 4 7; 1 1 0), (1 6 11; 0 1 1), (3 8 12; 1 0 1),
(2 9 11; 0 1 1), (1 5 8; 1 0 1), (6 7 12; 0 0 0), (3 4 10; 1 1 0),
(1 4 9 11; 1 0 0 1 1 0), (2 6 8 12; 1 0 0 1 1 0), (3 5 7 10; 1 0 0 1 1 0),
(2 5 7 11; 0 0 0 0 0 0), (3 4 9 12; 0 0 1 0 1 1), (1 6 8 10; 1 1 0 0 1 1),
(2 6 9 10; 0 1 0 1 0 1), (1 4 7 12; 0 1 0 1 0 1), (3 5 8 11; 0 0 1 0 1 1),
(1 5 9 12; 0 1 1 1 1 0), (2 4 8 10; 0 1 1 1 1 0), (3 6 7 11; 0 1 0 1 0 1).

The assertion follows by Theorem 1.17. □

Lemma 2.2. *There exist uniformly resolvable*

{3,4}-*URGDD of type* 9^4 *with* $r_3 = 12$ *and* $r_4 = 1$,
{3,4}-*URGDD of type* 9^4 *with* $r_3 = 9$ *and* $r_4 = 3$,
{3,4}-*URGDD of type* 9^4 *with* $r_3 = 6$ *and* $r_4 = 5$,
{3,4}-*URGDD of type* 9^4 *with* $r_3 = 3$ *and* $r_4 = 7$, *and*
4-RGDD *of type* 9^4 *with* ($r_3 = 0$ *and*) $r_4 = 9$.

Proof. The 4-RGDD of type 9^4 exists by Theorem 1.4. Let

$$\boldsymbol{G} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}.$$

A uniformly resolvable {3,4}-LRGDD$_3$ of type 3^4 with $r_3 = 12$ and $r_4 = 1$; each row forms a parallel class:

(1 4 10; 1 1 0), (5 8 12; 0 1 1), (3 7 11; 2 1 2), (2 6 9; 2 1 2),
(5 9 10; 1 0 2), (2 8 11; 1 2 1), (3 6 12; 1 2 1), (1 4 7; 0 0 0),
(2 7 12; 1 0 2), (3 4 10; 1 0 2), (6 8 11; 2 2 0), (1 5 9; 0 2 2),
(6 9 12; 0 2 2), (1 8 10; 1 2 1), (2 4 11; 0 0 0), (3 5 7; 1 1 0),
(3 5 11; 2 2 0), (2 4 8; 1 2 1), (1 7 12; 2 2 0), (6 9 10; 1 1 0),
(5 7 12; 2 0 1), (2 4 9; 2 2 0), (1 6 11; 1 1 0), (3 8 10; 1 1 0),
(4 7 10; 2 1 2), (1 5 8; 2 0 1), (3 9 11; 1 0 2), (2 6 12; 1 1 0),
(1 6 11; 2 0 1), (2 7 10; 2 0 1), (4 8 12; 0 0 0), (3 5 9; 0 0 0),
(1 5 10; 1 0 2), (4 7 11; 1 1 0), (2 6 8; 0 0 0), (3 9 12; 2 0 1),
(1 9 11; 1 2 1), (3 4 8; 0 2 2), (2 5 12; 0 2 2), (6 7 10; 2 2 0),
(1 8 12; 2 1 2), (3 6 7; 0 0 0), (2 5 10; 1 2 1), (4 9 11; 2 2 0),
(3 4 12; 2 1 2), (5 8 11; 2 1 2), (2 9 10; 0 1 1), (1 6 7; 0 1 1),
(2 5 7 11; 2 0 1 1 2 1), (3 6 8 10; 2 0 2 1 0 2), (1 4 9 12; 2 0 0 1 1 0).

A uniformly resolvable {3,4}-LRGDD_3 of type 3^4 with $r_3 = 9$ and $r_4 = 3$; each row forms a parallel class:

(6 8 11; 0 2 2), (2 7 10; 1 1 0), (1 5 12; 2 2 0), (3 4 9; 2 2 0),
(2 5 9; 2 1 2), (4 8 11; 0 0 0), (1 6 10; 0 0 0), (3 7 12; 1 2 1),
(1 4 11; 1 0 2), (5 8 10; 0 1 1), (3 9 12; 0 1 1), (2 6 7; 2 2 0),
(2 6 12; 1 2 1), (3 5 7; 0 2 2), (1 8 10; 2 2 0), (4 9 11; 1 1 0),
(6 9 12; 0 2 2), (3 8 10; 2 1 2), (1 4 7; 2 2 0), (2 5 11; 1 1 0),
(2 8 11; 1 2 1), (3 4 10; 1 0 2), (5 7 12; 0 2 2), (1 6 9; 2 1 2),
(3 6 10; 1 2 1), (2 9 11; 2 0 1), (1 5 8; 1 0 2), (4 7 12; 2 2 0),
(3 6 8; 2 0 1), (5 9 10; 1 2 1), (2 4 12; 0 1 1), (1 7 11; 1 1 0),
(3 5 11; 1 2 1), (6 7 10; 1 2 1), (2 4 8; 1 0 2), (1 9 12; 0 0 0),
(1 6 7 11; 1 0 2 2 1 2), (2 5 9 10; 0 0 0 0 0 0), (3 4 8 12; 0 1 0 1 0 2),
(2 6 8 12; 0 2 0 2 0 1), (3 5 7 11; 2 0 1 1 2 1), (1 4 9 10; 0 2 1 2 1 2),
(3 6 9 11; 0 1 0 1 0 2), (2 4 7 10; 2 0 2 1 0 2), (1 5 8 12; 0 1 1 1 1 0).

A uniformly resolvable {3,4}-LRGDD_3 of type 3^4 with $r_3 = 6$ and $r_4 = 5$; each row forms a parallel class:

(3 5 10; 1 2 1), (6 7 11; 1 0 2), (1 4 8; 0 1 1), (2 9 12; 1 2 1),
(3 8 11; 2 2 0), (2 4 10; 1 2 1), (1 6 9; 0 2 2), (5 7 12; 0 2 2),
(2 6 11; 1 0 2), (5 8 12; 1 0 2), (1 7 10; 1 2 1), (3 4 9; 1 2 1),
(3 5 7; 0 1 1), (1 4 12; 2 0 1), (6 8 10; 1 1 0), (2 9 11; 0 1 1),
(2 5 8; 1 1 0), (1 7 10; 0 0 0), (4 9 11; 2 1 2), (3 6 12; 1 0 2),
(1 5 11; 2 0 1), (3 8 12; 0 1 1), (2 6 7; 2 1 2), (4 9 10; 0 0 0),
(3 6 9 10; 0 1 0 1 0 2), (1 5 7 11; 0 2 2 2 2 0), (2 4 8 12; 0 0 0 0 0 0),
(2 6 7 12; 0 0 1 0 1 1), (3 5 9 10; 2 0 1 1 2 1), (1 4 8 11; 1 0 1 2 0 1),
(1 5 9 12; 1 0 2 2 1 2), (2 4 7 10; 2 2 1 0 2 2), (3 6 8 11; 2 1 0 2 1 2),
(2 5 8 10; 0 2 0 2 0 1), (3 4 7 11; 2 0 1 1 2 1), (1 6 9 12; 1 1 1 0 0 0),
(2 5 9 11; 2 2 2 0 0 0), (1 6 8 10; 2 2 1 0 2 2), (3 4 7 12; 0 2 2 2 2 0).

A uniformly resolvable {3,4}-LRGDD_3 of type 3^4 with $r_3 = 3$ and $r_4 = 7$; each row forms a parallel class:

(6 7 11; 2 0 1), (2 4 10; 1 0 2), (3 5 8; 0 1 1), (1 9 12; 1 1 0),
(2 8 11; 2 1 2), (3 6 12; 0 2 2), (1 4 7; 0 2 2), (5 9 10; 1 1 0),
(3 7 10; 2 2 0), (1 5 11; 0 0 0), (2 6 9; 2 1 2), (4 8 12; 1 1 0),
(1 6 8 12; 1 0 2 2 1 2), (2 4 9 11; 2 2 0 0 1 1), (3 5 7 10; 2 0 1 1 2 1),
(1 6 9 10; 2 2 0 0 1 1), (2 5 8 12; 2 1 2 2 0 1), (3 4 7 11; 1 1 0 0 2 2),
(3 5 9 12; 1 1 0 0 2 2), (1 4 8 11; 2 2 2 0 0 0), (2 6 7 10; 1 2 1 1 0 2),
(2 5 9 11; 1 0 2 2 1 2), (3 4 8 10; 0 2 0 2 0 1), (1 6 7 12; 0 0 0 0 0 0),
(2 6 8 10; 0 0 2 0 2 2), (1 5 7 11; 2 1 1 2 2 0), (3 4 9 12; 2 0 1 1 2 1),

(1 5 8 10; 1 1 1 0 0 0), (2 4 7 12; 0 1 0 1 0 2), (3 6 9 11; 1 2 2 1 1 0),
(1 4 9 10; 1 0 2 2 1 2), (2 5 7 12; 0 0 1 0 1 1), (3 6 8 11; 2 0 1 1 2 1).

The assertions follow by Theorem 1.17. □

Lemma 2.3. *There exist all admissible uniformly resolvable* $\{3, 4\}$*-URGDD of type* 12^4*,* $r_4 \in \{0, 2, 4, 6, 8, 10, 12\}$.

Proof. Let $\boldsymbol{G} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}$.

A uniformly resolvable {3,4}-LRGDD$_4$ of type 3^4 with $r_3 = 3$ and $r_4 = 10$; each row forms a parallel class:

(5 9 11; 2 2 0), (2 6 12; 0 3 3), (1 7 10; 1 1 0), (3 4 8; 2 2 0),
(2 8 11; 2 1 3), (4 7 12; 0 1 1), (3 5 10; 3 1 2), (1 6 9; 2 2 0),
(1 4 11; 1 2 1), (3 9 12; 1 1 0), (2 5 7; 1 2 1), (6 8 10; 0 1 1),
(2 4 9 12; 2 3 0 1 2 1), (1 5 7 10; 1 0 2 3 1 2), (3 6 8 11; 1 0 0 3 3 0),
(1 6 9 12; 0 3 1 3 1 2), (2 5 8 11; 3 1 3 2 0 2), (3 4 7 10; 3 1 0 2 1 3),
(1 6 7 10; 1 2 3 1 2 1), (3 4 9 11; 0 2 3 2 3 1), (2 5 8 12; 2 3 2 1 0 3),
(1 4 8 12; 2 1 2 3 0 1), (3 5 9 10; 0 0 3 0 3 3), (2 6 7 11; 1 0 2 3 1 2),
(1 5 8 12; 0 3 3 3 3 0), (3 6 7 11; 0 2 2 2 2 0), (2 4 9 10; 1 1 3 0 2 2),
(2 5 8 10; 0 0 0 0 0 0), (3 4 7 12; 1 0 0 3 3 0), (1 6 9 11; 3 0 3 1 0 3),
(3 6 8 12; 3 1 3 2 0 2), (2 4 9 10; 3 2 2 3 3 0), (1 5 7 11; 3 3 0 0 1 1),
(3 6 8 10; 2 3 2 1 0 3), (2 4 7 11; 0 1 0 1 0 3), (1 5 9 12; 2 1 0 3 2 3),
(3 5 7 12; 1 3 2 2 1 3), (1 4 8 11; 3 0 1 1 2 1), (2 6 9 10; 2 0 1 2 3 1),
(3 5 9 11; 2 3 1 1 3 2), (1 4 8 10; 0 2 0 2 0 2), (2 6 7 12; 3 3 1 0 2 2).

Therefore, there exists a $\{3, 4\}$-*URGDD* of type 12^4 with $r_3 = 3$ and $r_4 = 10$ by Theorem 1.17. The assertions follow by Lemma 1.15 □

Lemma 2.4. *There exists a uniformly resolvable* $\{3, 4\}$*-URGDD of type* 3^4 *with* $r_3 = 3$ *and* $r_4 = 1$.

Proof. There exists a 3-RGDD of type 4^3 with $r_3 = 4$ by Theorem 1.4. This is equivalent to the desired design. □

Lemma 2.5. *There exist all admissible uniformly resolvable* $\{3, 4\}$*-URGDD of type* 15^4*,* $r_4 \in \{1, 3, 5, 7, 9, 11, 13, 15\}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 3^4 with $r_4^0 = 1$ by Lemma 2.4. We expand all points of this design five times. The result is a $\{3, 4\}$-*URGDD* of type 15^4 with $r_4 = 5$. There exists a 4-RGDD of type 15^4 with $r_4 = 15$ by Theorem 1.4. There exists a $\{3, 4\}$-LRGDD$_5$ of type 3^4 with $r_4 \in \{1, 3, 7, 9, 11, 13\}$ in the online resource [34]. The assertions follow by Theorem 1.17. □

Lemma 2.6. *There exists a uniformly resolvable* $\{3, 4\}$*-URGDD of type* 18^4*,* $r_4 \in \{0, 2, \ldots, 18\}$.

Proof. There exists 4-RGDD of type 18^4 by Theorem 1.4. There exists a uniformly resolvable {3,4}-*URGDD* of type 6^4 with $r_4 \in \{0, 2, 4\}$ by Lemma 2.1. We expand each point three times, use a {3,4}-*URGDD* of type 3^4 with $r_3 = 3$, $r_4 = 1$ (Lemma 2.4) as ingredient design, and obtain a {3,4}-*URGDD* of type 18^4 with $r_4 \in \{0, 2, 4\}$. There exists a {3, 4}-LRGDD$_6$ of type 3^4 with $r_4 \in \{6, 8, 10, 12, 14, 16\}$ in the online resource [34]. The assertions follow by Theorem 1.17. □

Lemma 2.7. *There exists a uniformly resolvable* {3,4}*-URGDD of type* 21^4*,* $r_4 \in \{1, 3, \ldots, 21\}$.

Proof. There exists a {3, 4}-URGDD of type 3^4 with $r_4^0 = 1$ by Lemma 2.4. We expand all points of this design seven times. The result is a {3,4}-*URGDD* of type 21^4 with $r_4 = 7$. There exists a 4-RGDD of type 21^4 with $r_4 = 21$ by Theorem 1.4.

There exists a {3, 4}-LRGDD$_7$ of type 3^4 with $r_4 \in \{1, 3, 5, 9, 11, 13, 15, 17, 19\}$ in the online resource [34]. The assertions follow by Theorem 1.17. □

Lemma 2.8. *There exists a uniformly resolvable* {3,4}*-URGDD of type* 27^4*,* $r_4 \in \{1, 3, 5, 7, 9, 27\}$.

Proof. There exists a uniformly resolvable {3,4}-*URGDD* of type 9^4 with $r_4 \in \{1, 3, 5, 7, 9\}$ by Lemma 2.2. We expand each point three times, use a {3,4}-*URGDD* of type 3^4 with $r_3 = 3$, $r_4 = 1$ (Lemma 2.4) as ingredient design, and obtain a {3,4}-*URGDD* of type 27^4 with $r_4 \in \{1, 3, 5, 7, 9\}$. There exists a 4-RGDD of type 27^4 by Theorem 1.4. □

Lemma 2.9. *There exists a uniformly resolvable* {3,4}*-URGDD of type* 6^6*,* $r_4 \in \{0, 2, 4, 6, 8\}$.

Proof. There exists a 3-RGDD of type 6^6 by Theorem 1.4. All other designs are constructed directly in [34]. □

Lemma 2.10. *There exists a* {3, 4}*-URGDD of type* 60^4*,* $r_4 \in \{0, 2, \ldots, 60\}$.

Proof. There exists a 4-RGDD of type 5^4 by Theorem 1.4, which is our master design. We take all designs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times and obtain a {3, 4}-*URGDD* of type 60^4 with $r_4 \in \{0, 2, \ldots, 60\}$. □

Lemma 2.11. *There exists a uniformly resolvable URD*({3, 4}; 276) *with* $r_4 = 9$.

Proof. There exists a perfectly uniform semiframe {3, 4}-LRGDD$_{69}$ of type 1^4 with $\tilde{r}_3 = 30$ per group and $r_4 = 9$ in the online resource [34]. This results in a perfectly uniform semiframe {3, 4}-SF of type 69^4 with $\tilde{r}_3 = 30$ per group and $r_4 = 9$ by Theorem 1.19. We fill the groups with a 3-RGDD of type 1^{69} with $r_3 = 34$ (Theorem 1.4). Therefore, we obtain a URD({3, 4}; 276) with $r_4 = 9$. □

Theorem 2.12. *There exists a URD*({3, 4}; v) *with* $r_4 = 9$ *if and only if* $v \equiv 0 \pmod{12}$ *except* $v = 12, 24$.

Proof. A URD($\{3, 4\}$; 276) with $r_4 = 9$ is obtained in Lemma 2.11. The assertion follows by Theorem 1.12. □

3. ADMISSIBLE URDs FOR SMALL v

Lemma 3.1. *There exist all admissible URD*($\{3, 4\}$; 24)*,* $r_4 \in \{1, 3, 5, 7\}$.

Proof. The assertion follows by Theorem 1.12. □

Lemma 3.2. *There exist all admissible URD*($\{3, 4\}$; 36)*,* $r_4 \in \{1, 3, 5, 7, 9, 11\}$.

Proof. There exists a 4-RGDD of type 3^{12} with $r_4 = 11$ by Theorem 1.4. The assertion follows by Theorem 1.12. □

Lemma 3.3. *There exist all admissible URD*($\{3, 4\}$; v)*,* $v \in \{48, 96, 144, 192\}$.

Proof. The assertion follows by Theorem 1.14. □

Lemma 3.4. *There exist all admissible URD*($\{3, 4\}$; 60)*,* $r_4 \in \{1, 3, 5, \ldots, 19\}$.

Proof. There exists a uniformly resolvable $\{3, 4\}$-URGDD of type 15^4 with $r_4 \in \{11, 13\}$ by Lemma 2.5. Filling the groups with a 3-RGDD of type 1^{15} results in a URD ($\{3, 4\}$; 60) with $r_4 \in \{11, 13\}$. The assertion follows by Theorems 1.12 and 1.13. □

Lemma 3.5. *There exist all admissible URD*($\{3, 4\}$; 72)*,* $r_4 \in \{1, 3, 5, \ldots, 23\}$.

Proof. There exists a $\{3, 4\}$-LRGDD$_3$ of type 1^{24} with $r_4 \in \{11, 13, 15\}$ in the online resource [34]. Therefore, there exists a URD($\{3, 4\}$; 72) with $r_4 \in \{11, 13, 15\}$ by Theorem 1.17. There exists a $\{3, 4\}$-LRGDD$_6$ of type 2^6 with $r_4 \in \{16, 18\}$ in the online resource [34]. Therefore, there exists a $\{3, 4\}$-URGDD of type 12^6 with $r_4 \in \{16, 18\}$ by Theorem 1.17. Filling the groups with a $\{3, 4\}$-URGDD of type 1^{12} with $r_3 = 4$, $r_4 = 1$ (Lemma 2.4) results in a URD($\{3, 4\}$; 72) with $r_4 \in \{17, 19\}$. The assertion follows by Theorems 1.12 and 1.13. □

Lemma 3.6. *There exist all admissible URD*($\{3, 4\}$; 84)*,* $r_4 \in \{1, 3, 5, \ldots, 21, 25, 27\}$*, possibly excepting* $r_4 = 23$.

Proof. There exists a labeled perfectly uniform semiframe $\{3, 4\}$-LRGDD$_{21}$ of type 1^4 with $\tilde{r}_3 \in \{5, 4, 3, 2, 1\}$ per group and $r_4 \in \{11, 13, 15, 17, 19\}$, respectively, in the online resource [34]. This results in a semiframe $\{3, 4\}$-SF of type 21^4 with $\tilde{r}_3 \in \{5, 4, 3, 2, 1\}$ per group and $r_4 \in \{11, 13, 15, 17, 19\}$, respectively, by Theorem 1.19. We fill the groups with a 3-RGDD of type 1^{21} with $r_3 = 10$ (Theorem 1.4). This expands all partial 3-pc and induces additional 3-pc. Therefore, we obtain a URD($\{3, 4\}$; 84) with $r_4 \in \{11, 13, 15, 17, 19\}$.There exists a 4-RGDD of type 21^4 with $r_4 = 21$ by Theorem 1.4. Filling the groups with a 3-RGDD of type 1^{21} (Theorem 1.4) results in a URD($\{3, 4\}$; 84) with $r_4 = 21$. The assertion follows by Theorems 1.12 and 1.13. □

Lemma 3.7. *There exist all admissible URD*$(\{3, 4\}; 108)$, $r_4 \in \{1, 3, 5, \ldots, 27, 33, 35\}$, *possibly excepting* $r_4 \in \{29, 31\}$.

Proof. There exists a labeled perfectly uniform semiframe $\{3, 4\}$-LRGDD$_{27}$ of type 1^4 with $\tilde{r}_3 \in \{8, 7, 6, 5, 4, 3, 2, 1\}$ per group and $r_4 \in \{11, 13, 15, 17, 19, 21, 23, 25\}$, respectively, in the online resource [34]. This results in a semiframe $\{3, 4\}$-SF of type 27^4 with $\tilde{r}_3 \in \{8, 7, 6, 5, 4, 3, 2, 1\}$ per group and $r_4 \in \{11, 13, 15, 17, 19, 21, 23, 25\}$, respectively, by Theorem 1.19. We fill the groups with a 3-RGDD of type 1^{27} with $r_3 = 13$ (Theorem 1.4). This expands all partial 3-pc and induces additional 3-pc. Therefore, we obtain a URD$(\{3, 4\};\ 108)$ with $r_4 \in \{11, 13, 15, 17, 19, 21, 23, 25\}$. There exists a 4-RGDD of type 27^4 with $r_4 = 27$ by Theorem 1.4. Filling the groups with a 3-RGDD of type 1^{27} (Theorem 1.4) results in a URD$(\{3, 4\};\ 108)$ with $r_4 = 27$. The assertion follows by Theorems 1.12 and 1.13. □

Lemma 3.8. *There exists a* $\{3, 4\}$-*URGDD of type* 24^5 *for* $r_4 \in \{0, 2, 4, \ldots, 16, 32\}$.

Proof. There exists a 4-RGDD of type 5^4 by Theorem 1.4. This is also a $\{4, 5\}$-URGDD of type 4^5 $r_4 = 4$, $r_5 = 1$, which we take as the master design. There exist a 3-RGDD of types 6^4 and 6^5 by Theorem 1.4 and a $\{3,4\}$-URGDD of type 6^4 with $r_4 \in \{0, 2, 4\}$ by Lemma 2.1, which are our ingredient designs. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 24^5 with $r_4 \in \{0, 2, 4, \ldots, 16\}$, as we fill all parallel classes appropriately. There exists a 4-RGDD of type 24^5 with $r_4 = 32$ by Theorem 1.4. □

Lemma 3.9. *There exist all admissible URD*$(\{3, 4\};\ 120)$, *possibly excepting* $r_4 \in \{27, 29, 31\}$.

Proof. We fill each group in Lemma 3.8 with the same appropriate URD$(\{3, 4\};\ 24)$ and obtain a URD$(\{3, 4\};\ 120)$ with $r_4 \in \{1, 3, 5, \ldots, 23, 33, 35, 37, 39\}$.

There exists a $\{3, 4\}$-URGDD of type 3^8, $r_4^0 = 5$ by Lemma 3.1, which we take as the master design. There exist a 3-RGDD of type 5^3 and a 4-RGDD of type 5^4 by Theorem 1.4, which are our ingredient designs. We expand all points of the master design five times. We obtain a $\{3, 4\}$-URGDD of type 15^8 with $r_4 = 25$, filling the groups with a 3-RGDD of type 1^{15}(Theorem 1.4) results in a URD$(\{3, 4\};\ 120)$ with $r_4 = 25$. □

Lemma 3.10. *There exist all admissible URD*$(\{3, 4\}; 132)$, *possibly excepting* $r_4 \in \{35, 37, 39\}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 3^4 with $r_3^0 = 3$, $r_4^0 = 1$ by Lemma 2.4, which we take as the master design. There exist a 3-RGDD of type 11^3 and a 4-RGDD of type 11^4 by Theorem 1.4, which are our ingredient designs. We expand all points of the master design 11 times. The 4-pc of the master design results in 11 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 33^4 with $r_4 = 11$, filling the groups with a 3-RGDD of type 1^{33} (Theorem 1.4) results in a URD$(\{3, 4\};\ 132)$ with $r_4 = 11$.

There exists a labeled perfectly uniform semiframe $\{3,4\}$-LRGDD$_{33}$ of type 1^4 with $\tilde{r}_3 \in \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$ per group and $r_4 \in \{13, 15, 17, 19, 21, 23,$

$25, 27, 29, 31\}$, respectively, in the online resource [34]. This results in a perfectly uniform semiframe $\{3, 4\}$-SF of type 33^4 with $\tilde{r}_3 \in \{10, 9, 8, 7, 6, 5, 4, 3, 2, 1\}$ per group and $r_4 \in \{13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$, respectively, by Theorem 1.19. We fill the groups with a 3-RGDD of type 1^{33} with $r_3 = 16$ (Theorem 1.4). This expands all partial 3-pc and induces additional 3-pc. Therefore, we obtain a URD($\{3, 4\}$; 132) with $r_4 \in \{13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$.

There exists a 4-RGDD of type 33^4 with $r_4 = 33$ by Theorem 1.4. Filling the groups with a 3-RGDD of type 1^{33} (Theorem 1.4) results in a URD($\{3, 4\}$; 132) with $r_4 = 33$. The assertion follows by Theorems 1.12 and 1.13. □

Lemma 3.11. *There exist all admissible URD*($\{3, 4\}$; 156)*, possibly excepting* $r_4 \in \{41, 43, 45, 47\}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 3^4 with $r_3^0 = 3$, $r_4^0 = 1$ by Lemma 2.4, which we take as the master design. There exist a 3-RGDD of types 13^3 and a 4-RGDD of type 13^4 by Theorem 1.4, which are our ingredient designs. We expand all points of the master design 13 times. The 4-pc of the master design results in 13 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 39^4 with $r_4 = 13$, filling the groups results in a URD($\{3, 4\}$; 156) with $r_4 = 13$.

There exists a 3-RGDD of type 52^3 by Theorem 1.4. Filling the groups with a 4-RGDD of type 1^{52} (Theorem 1.4) results in a URD($\{3, 4\}$; 156) with $r_4 = 17$.

There exists a labeled perfectly uniform semiframe $\{3, 4\}$-LRGDD$_{39}$ of type 1^4 with $\tilde{r}_3 \in \{14, 12, 10, 9, 8, \ldots, 1\}$ per group and $r_4 \in \{11, 15, 19, 21, \ldots, 37\}$, respectively, in the online resource [34]. This results in a perfectly uniform semiframe$\{3, 4\}$-SF of type 39^4 with the same $\tilde{r}_3$ per group and r_4, respectively, by Theorem 1.19. We fill the groups with a 3-RGDD of type 1^{39} with $r_3 = 19$ (Theorem 1.4). This expands all partial 3-pc and induces additional 3-pc. Therefore, we obtain a URD($\{3, 4\}$; 156) with $r_4 \in \{11, 15, 19, 21, \ldots, 37\}$.

There exists a 4-RGDD of type 39^4 with $r_4^0 = 39$ by Theorem 1.4. Filling the groups with a 3-RGDD of type 1^{39} (Theorem 1.4) results in a URD($\{3, 4\}$; 156) with $r_4 = 39$. The assertion follows by Theorems 1.12 and 1.13. □

Lemma 3.12. *There exists a* $\{3, 4\}$*-URGDD of type* 36^g *with* $r_4 \in \{0, g-1, g+1, \ldots, 9(g-1)\}$ *for* $g \geq 4$*, g odd.*

Proof. Let $g \geq 4$ odd. There exists a 3-RGDD of type 36^g by Theorem 1.4.

There exists a 4-RGDD of type g^4 by Theorem 1.4. This is also a $\{4, g\}$-URGDD of type 4^g, $r_4 = g - 1$, $r_g = 1$, which we take as the master design. We take the URGDDs of Lemma 2.2 as ingredient designs. We expand all points of the master design nine times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 1, 3, 5, 7, or 9 4-pcs. There exists a 3-RGDD of type 9^g by Theorem 1.4 for g odd. We obtain a $\{3, 4\}$-URGDD of type 36^g with $r_4 \in \{0, g-1, g+1, \ldots, 9(g-1)\}$, as we fill all parallel classes appropriately. □

Lemma 3.13. *There exists a* $\{3, 4\}$*-URGDD of type* 36^{3i+1} *for* $i \geq 1$ *and* $r_4 \in \{0, 4i, 4i+2, 4i+4, \ldots, 36i\}$.

Proof. There exists a 3-RGDD of type 36^{3i+1} for $i \geq 1$by Theorem 1.4.

There exists a 4-RGDD of type 4^{3i+1} for $i \geq 1$by Theorem 1.4, which we take as the master design and all designs of Lemma 2.2 as ingredient designs. We expand all points of the master design nine times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 1, 3, 5, 7, or 9 4-pcs. We obtain a {3, 4}-URGDD of type 36^{3i+1} with $r_4 \in \{4i, 4i+2, 4i+4, \ldots, 36i\}$, as we fill all parallel classes appropriately. □

Lemma 3.14. *There exists a* $\{3, 4\}$*-URGDD of type* 24^{3i+1} *for* $i \geq 1$ *and* $r_4 \in \{0,\ 2,\ 4, \ldots, 16i,\quad 24i\}$.

Proof. There exists a 4-RGDD of type 24^{3i+1} for $i \geq 1$ by Theorem 1.4.

There exists a 4-RGDD of type 4^{3i+1} for $i \geq 1$ by Theorem 1.4, which we take as the master design. We take the URGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a {3, 4}-URGDD of type 24^{3i+1} with $r_4 \in \{0,\ 2,\ 4, \ldots, 16i\}$, as we fill all parallel classes appropriately. □

Lemma 3.15. *There exists a* $\{3, 4\}$*-URGDD of type* 24^{3i+1} $r_4 \in \{0, 2, 4, \ldots, 22i,\ 24i\}$ *for* $i \geq 2$ *and* $i \notin \{3, 11, 15\}$.

Proof. There exists an RTD(6, 3*i*+1) for $i \geq 2$, $i \notin \{3, 7, 11, 15\}$ by Theorem 1.3 and therefore also a {6, 3*i*+1}-URGDD of type 6^{3i+1} with $r_6 = 3i$ and $r_{3i+1} = 1$.

We apply the last as the master design. There exist a {3, 4}-URGDD of type 4^6, $r_4 \in \{0,\ 2,\ 4,\ 6\}$ by Lemma 3.1 and a 4-RGDD of type 4^{3i+1} with $r_4^0 = 4i$ by Theorem 1.4, which we take as ingredient designs. We expand all points of the master design four times. All blocks of any parallel class have to be filled with the same ingredient design. Each 6-pc of the master design results in 0, 2, 4, or 6 4-pcs. We obtain a {3, 4}-URGDD of type 24^{3i+1} with $r_4 \in \{4i,\ 4i+2, \ldots, 4i+18i\}$, as we fill all parallel classes appropriately. The assertion follows by Lemma 3.14.

There exists a 4-RGDD of type 2^{22} by Theorem 1.4, which we take as the master design. We take the URGDDs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 2, ..., or 12 4-pcs. We obtain a {3, 4}-URGDD of type 24^{22} with $r_4 \in \{0,\ 2,\ 4, \ldots, 168\}$, as we fill all parallel classes appropriately. □

Lemma 3.16. *There exist all admissible URD*$(\{3, 4\};\ 168)$.

Proof. There exists a $\{3, 4\}$-URGDD of type 24^7 with $r_4 \in \{0, 2, 4, \ldots, 44, 48\}$ by Lemma 3.15. The assertion follows by filling all groups appropriately with the same URD$(\{3, 4\};\ 24)$. □

Lemma 3.17. *There exist all admissible URD*$(\{3, 4\};\ 180)$.

Proof. There exists a $\{3,4\}$-URGDD of type 36^5 with $r_4 \in \{0,4,6,8,\ldots,36,48\}$ by Lemma 3.12 and Theorem 1.4. The assertion follows by filling all groups appropriately with the same URD($\{3,4\}$; 36) (Lemma 3.2). □

Lemma 3.18. *There exist all admissible* $\{3,4\}$*-URGDD of type* 36^4. *There exists a* $\{3,4\}$*-URGDD of type* 36^6 *with* $r_4 \in \{0,2,\ldots,54,60\}$.

Proof. There exists a $\{3,4\}$-URGDD of type 3^4 with $r_4 = 1$ by Lemma 2.4. We expand all points of this design 12 times and obtain a $\{3,4\}$-URGDD of type 36^4 with $r_4 = 2$. For $u = 4$, the assertion follows by Lemma 3.13.

There exists a $\{3,4\}$-URGDD of type 4^6 with $r_4 \in \{0, 2, 4, 6\}$ by Lemma 3.1, which we take as the master design. We take the RGDDs of Lemma 2.2 and the 3-RGDD of type 9^3 (Theorem 1.4) as ingredient designs. We expand all points of the master design nine times. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 1, 3, 5, 7, or 9 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 36^6 with $r_4 \in \{0,2,\ldots,54\}$. There exists a 4-RGDD of type 36^6 with $r_4 = 60$ by Theorem 1.4. □

Lemma 3.19. *There exist all admissible URD*($\{3,4\}$; 216).

Proof. There exists a $\{3,4\}$-URGDD of type 36^6 with $r_4 \in \{0,2,\ldots,54,60\}$ by Lemma 3.18. The assertion follows by filling all groups appropriately with the same URD($\{3,4\}$; 36). □

Lemma 3.20. *There exists a* $\{3,4\}$*-URGDD of type* 36^u*,* $r_4 \in \{0, 2,\ldots,8(u-1)\}$ *for* $u \geq 7$.

Proof. There exists a 6-RGDD of type u^6, $u \geq 7$, and $u \notin \{10,14,15,18,20,22,26,30,34,38,46,60\}$ by Theorem 1.3. This is also a $\{6, u\}$-URGDD of type 6^u $r_6 = u-1$, $r_u = 1$, which we take as the master design. We take the URGDDs of Lemma 2.9 as ingredient designs. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in $0,2,\ldots,8$ 4-pcs. There exists a 3-RGDD of type 6^u by Theorem 1.4. We obtain a $\{3, 4\}$-URGDD of type 36^u with $r_4 \in \{0, 2,\ldots,8(u-1)\}$, as we fill all parallel classes appropriately.

There exists a 4-RGDD of type 6^u with $r_4 = 2(u-1)$ for $u \in \{10,14,18,20,22,26,30,34,38,46,60\}$ by Theorem 1.4, which we take as the master design. We take the URGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 36^u with $r_4 \in \{0, 2, 4,\ldots,8(u-1)\}$, as we fill each parallel class appropriately.

There exists a $\{3,4\}$-URGDD of type 4^{15} with $r_4 \in \{0,2,\ldots,18\}$ by Lemma 3.4. We expand all points of this design nine times and obtain a $\{3, 4\}$-URGDD of type 36^{15} with $r_4 \in \{0,2,\ldots,162\}$ by filling in with the $\{3,4\}$-URGDD of type 9^4 from Lemma 2.2. □

Lemma 3.21. *There exists a* $\{3,4\}$*-frame of type* 180^u *for* $u \geq 5$ *and* $\tilde{r}_4 \in \{4, 6, 8, \ldots, 60\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 12^u for $u \geq 5$ with $\widehat{r_4} = 4$ per group by Theorem 1.6, which we take as the master design. We take the RGDDs of Lemma 2.5 as ingredient designs. We expand all points of the master design 15 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 1, 3, 5, ... , 15 4-pcs. We obtain a $\{3, 4\}$-frame of type 180^u with $\tilde{r}_4 \in \{4, 6, 8, \ldots, 60\}$ per group of the frame. □

Lemma 3.22. *For* $i \geq 5$*, there exists all admissible* $\{3,4\}$*-URGDD of type* 36^{5i+1}*, possibly excepting* $r_4 \in \{60i - 4, 60i - 2\}$.

Proof. There exists a $\{3,4\}$-frame of type 180^u for $u \geq 5$ and $\tilde{r}_4 \in \{4, 6, 8, \ldots, 60\}$ per group of the frame by Lemma 3.21. There exists a $\{3,4\}$-URGDD of type 36^6 with $r_4 \in \{0, 2, 4, \ldots, 54, 60\}$ by Lemma 3.18. Adjoin 36 infinite points to the frame and fill each group with the above URGDD, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. Then the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. The result is a $\{3,4\}$-URGDD of type 36^{5i+1} for $i \geq 5$ and $r_4 \in \{4i, 4i+2, \ldots, 60(i-1)+54, 60i\}$. The assertion follows by Lemma 3.20. □

Lemma 3.23. *There exist all admissible* $\{3,4\}$*-URGDD of type* 36^u *for* $u \in \{7, 8, 24\}$.

Proof. For $u = 7$, the assertion follows by Lemmas 3.20 and 3.13.

There exists a $\{3,4\}$-URGDD of type 3^8 with $r_4 \in \{1, 3, 5, 7\}$ by Lemma 3.1. There exists a $\{3,4\}$-URGDD of type 3^{24} with $r_4 \in \{1, 3, \ldots, 23\}$ by Lemma 3.5.

We expand all points of each design 12 times, taking the RGDDs of Lemma 2.3 as ingredient designs to obtain a $\{3,4\}$-URGDD of type 36^8 with $r_4 \in \{0, 2, \ldots, 84\}$ and a $\{3, 4\}$-URGDD of type 36^{24} with $r_4 \in \{0, 2, \ldots, 276\}$, respectively. □

Lemma 3.24. *There exists a* $\{3,4\}$*-URGDD of type* 36^{17} *for* $r_4 \in \{0, 2, 4, \ldots, 176, 192\}$.

Proof. There exists an RTD(12, 17) by Theorem 1.3. Therefore, there exists a $\{12, 17\}$-URGDD of type 12^{17} with $r_{12} = 16$ and $r_{17} = 1$. We apply the latter as the master design. There exist a $\{3, 4\}$-URGDD of type 3^{12}, $r_4 \in \{1, 3, 5, 7, 9, 11\}$ by Lemma 3.2 and a 3-RGDD of type 3^{17} by Theorem 1.4, which we take as ingredient designs. We expand all points of the master design three times. All blocks of any parallel class have to be filled with the same ingredient design. Each 12-pc of the master design results in 1, 3, ..., or 11 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 36^{17} with $r_4 \in \{16, 18, \ldots, 176\}$, as we fill all parallel classes appropriately. The assertion follows by Lemma 3.20 and Theorem 1.4. □

Lemma 3.25. *There exist all admissible URD*$(\{3, 4\}; 300)$.

Proof. There exist a 3-RGDD of type 60^5 and a 4-RGDD of type 60^5 by Theorem 1.4.

There exists a 4-RGDD of type 5^4 by Theorem 1.4. This is also a $\{4, 5\}$-URGDD of type 4^5 $r_4 = 4$, $r_5 = 1$, which we take as the master design. We take the URGDDs of Lemma 2.5 and a 3-RGDD of type 15^5 (Theorem 1.4) as ingredient designs. We expand all points of the master design 15 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in $1, 3, \ldots, 15$ 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 60^5 with $r_4 \in \{0, 4, 6, \ldots, 60, 80\}$, as we fill all parallel classes appropriately. By filling all groups appropriately with the same URD$(\{3, 4\};\ 60)$ (Lemma 3.4), we obtain all admissible URD$(\{3, 4\};\ 300)$. □

Lemma 3.26. *There exist all admissible* URD$(\{3, 4\}; v)$ *for* $v \in \{252, 360, 468\}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 36^{3i+1}, $r_4 \in \{0,\ 2,\ 4, \ldots, 36i\}$ for $i \in \{2, 3, 4\}$ by Lemmas 3.13 and 3.20. Filling in all groups with the same appropriate URD $(\{3, 4\};\ 36)$ results in all admissible URD$(\{3, 4\}; v)$ for $v \in \{252, 360, 468\}$. □

We summarize the results of this section about small URDs.

Theorem 3.27. *There exist all admissible URD*$(\{3, 4\};\ v)$*,* $v \equiv 0 \pmod{12}$*,* $v < 200$*, except when* $v = 12$ *and* $r_4 = 3$ *and possibly excepting:*

$v = 84$*:* $r_4 = 23$*;*

$v = 108$*:* $r_4 \in \{29, 31\}$*;*

$v = 120$*:* $r_4 \in \{27, 29, 31\}$*;*

$v = 132$*:* $r_4 \in \{35, 37, 39\}$*;*

$v = 156$*:* $r_4 \in \{41, 43, 45, 47\}$.

Proof. The assertion follows by the lemmas of this section. □

4. SOME {3, 4}-URGDDS AND {3, 4}-FRAMES

Lemma 4.1. *There exists a* $\{3, 4\}$*-frame of type* 108^u *for* $u \geq 5$*,* $u \notin \{15,\ 23,\ 27\}$*,* $u \equiv 1 \pmod 2$ *and* $\tilde{r}_4 \in \{0,\ 2,\ 4, \ldots, 24\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 18^u for $u \geq 5$, $u \notin \{15,\ 23,\ 27\}$, $u \equiv 1 \pmod 2$ with $\widehat{r}_4 = 6$ per group by Theorem 1.6, which we take as the master design. We take the URGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any holey parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a $\{3, 4\}$-frame of type 108^u with $\tilde{r}_4 \in \{0,\ 2,\ 4, \ldots, 24\}$ per group of the frame. □

Lemma 4.2. *There exists a* $\{3, 4\}$*-frame of type* 108^u *for* $u \geq 5$ *and* $\tilde{r}_4 \in \{4, 6, 8, \ldots, 36\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 12^u for $u \geq 5$ with $\widehat{r}_4 = 4$ per group by Theorem 1.6, which we take as the master design. We take the RGDDs of Lemma 2.2 as ingredient designs. We expand all points of the master design nine times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 1, 3, 5, 7, or 9 4-pcs. We obtain a {3, 4}-frame of type 108^u with $\tilde{r}_4 \in \{4, 6, 8, \ldots, 36\}$ per group of the frame. □

Lemma 4.3. *There exists a* $\{3, 4\}$*-frame of type* 144^u *for* $u \geq 5$ *and* $\tilde{r}_4 \in \{0, 2, 4, \ldots, 48\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 12^u for $u \geq 5$ with $\widehat{r}_4 = 4$ per group by Theorem 1.6, which we take as the master design. We take the RGDDs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 0, 2, 4, 6, 8, 10, or 12 4-pcs. We obtain a {3, 4}-frame of type 144^u with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 48\}$ per group of the frame. □

Lemma 4.4. *There exists a* $\{3, 4\}$*-frame of type* 216^u *for* $u \geq 5$ *and* $\tilde{r}_4 \in \{0, 2, 4, \ldots, 72\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 12^u for $u \geq 5$ with $\widehat{r}_4 = 4$ per group by Theorem 1.6, which we take as the master design. We take the URGDDs of Lemma 2.6 as ingredient designs. We expand all points of the master design 18 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 0, 2, ..., or 18 4-pcs. We obtain a {3, 4}-frame of type 216^u with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 72\}$ per group of the frame. □

Lemma 4.5. *There exists a* $\{3, 4\}$*-frame of type* 36^{2i+1} *for* $i \geq 2$*,* $i \notin \{3, 11, 13, 17, 19, 23\}$ *and* $\tilde{r}_4 \in \{0, 2, 4, 6, 8\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 6^{2i+1} with $\widehat{r}_4 = 2$ per group for $i \geq 2$ and $i \notin \{3, 11, 13, 17, 19, 23\}$ by Theorem 1.6, which we take as the master design. We take the URGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any holey parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a {3, 4}-frame of type 36^{2i+1} with $\tilde{r}_4 \in \{0, 2, 4, 6, 8\}$ per group of the frame. □

Lemma 4.6. *There exists a* $\{3, 4\}$*-URGDD of type* 180^u *for* $u \geq 4$ *and* $r_4 \in \{0,\ 4(u-1), 4(u-1)+2, \ldots, 60(u-1)\}$.

Proof. There exists a 3-RGDD of type 180^u for $u \geq 4$ by Theorem 1.4.

There exists a 4-RGDD of type 12^u for $u \geq 4$ by Theorem 1.4, which we take as the master design and all designs of Lemma 2.5 as ingredient designs. We expand all points of the master design 15 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in $1, 3, 5, \ldots, 15$ 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 180^u with $r_4 \in \{0, 4(u-1), 4(u-1)+2, \ldots, 60(u-1)\}$, as we fill all parallel classes appropriately. □

Lemma 4.7. *There exists a* $\{3, 4\}$-*frame of type* 360^u *for* $u \geq 5$ *and* $\tilde{r}_4 \in \{0, 2, 4, \ldots, 80\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 60^u for $u \geq 5$ with $\widehat{r}_4 = 20$ per group by Theorem 1.6, which we take as the master design. We take the URGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any holey parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a $\{3, 4\}$-frame of type 360^u with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 80\}$ per group of the frame. □

Lemma 4.8. *There exists a* $\{3, 4\}$-*frame of type* 360^u *for* $u \geq 5$ *and* $\tilde{r}_4 \in \{8, 10, 12, \ldots, 120\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 24^u for $u \geq 5$ with $\widehat{r}_4 = 8$ per group by Theorem 1.6, which we take as the master design. We take the RGDDs of Lemma 2.5 as ingredient designs. We expand all points of the master design 15 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in $1, 3, 5, \ldots, 15$ 4-pcs. We obtain a $\{3, 4\}$-frame of type 360^u with $\tilde{r}_4 \in \{8, 10, 12, \ldots, 120\}$ per group of the frame. □

Lemma 4.9. *There exists a* $\{3, 4\}$-*URGDD of type* 180^{2i} *for* $i \geq 2$ *and* $r_4 \in \{0, 2, 4, \ldots, 40(2i-1)\}$.

Proof. There exists a 4-RGDD of type 30^{2i} for $i \geq 2$ by Theorem 1.4, which we take as the master design. We take the URGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 180^{2i} with $r_4 \in \{0, 2, 4, \ldots, 40(2i-1)\}$, as we fill all parallel classes appropriately. □

Theorem 4.10. *There exists a* $\{3, 4\}$-*URGDD of type* 180^{2i} *for* $i \geq 2$ *and* $r_4 \in \{0, 2, 4, \ldots, 60(2i-1)\}$.

Proof. The assertion follows by Lemmas 4.6 and 4.9. □

Lemma 4.11. *There exists a* $\{3,4\}$*-URGDD of type* 120^{3i+1} *for* $i \geq 1$ *and* $r_4 \in \{0, 2, 4, \ldots, 80i\}$.

Proof. There exists a 4-RGDD of type 20^{3i+1} with $r_4^0 = 20i$ for $i \geq 1$ by Theorem 1.4, which we take as the master design. We take the URGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 120^{3i+1} with $r_4 \in \{0, 2, 4, \ldots, 80i\}$, as we fill all parallel classes appropriately. □

Lemma 4.12. *There exists a* $\{3,4\}$*-URGDD of type* 120^{3i+1} *for* $i \geq 1$ *and* $r_4 \in \{8i, 8i+2, \ldots, 120i\}$.

Proof. There exists a 4-RGDD of type 8^{3i+1} with $r_4^0 = 8i$ for $i \geq 1$ by Theorem 1.4, which we take as the master design and all designs of Lemma 2.5 as ingredient designs. We expand all points of the master design 15 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in $1, 3, \ldots, 15$ 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 120^{3i+1} with $r_4 \in \{8i, 8i+2, \ldots, 120i\}$, as we fill all parallel classes appropriately. □

Theorem 4.13. *There exists a* $\{3,4\}$*-URGDD of type* 120^{3i+1} *for* $i \geq 1$ *and* $r_4 \in \{0, 2, 4, \ldots, 120i\}$.

Proof. The assertion follows by Lemmas 4.11 and 4.12. □

Lemma 4.14. *There exists a* $\{3,4\}$*-URGDD of type* 60^{3i+1} *for* $i \geq 1$ *and* $r_4 \in \{0, 4i, 4i+2, \ldots, 60i\}$.

Proof. There exists a 3-RGDD of type 60^{3i+1} for $i \geq 1$ by Theorem 1.4.

There exists a 4-RGDD of type 4^{3i+1} with $r_4^0 = 4i$ for $i \geq 1$ by Theorem 1.4, which we take as the master design and all designs of Lemma 2.5 as ingredient designs. We expand all points of the master design 15 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in $1, 3, \ldots, 15$ 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 60^{3i+1} with $r_4 \in \{4i, 4i+2, \ldots, 60i\}$, as we fill all parallel classes appropriately. □

Lemma 4.15. *There exist* $\{3,4\}$*-URGDDs of type* 12^i *for*

$i = 5$, $r_4 \in \{0, 4, 16\}$;

$i = 10$, $r_4 \in \{0, 12, 36\}$;

$i = 325$, $r_4 \in \{1,056, 1,058, \ldots, 1,296\}$.

Proof. There exist a 3-RGDD of type 12^i and 4-RGDD of type 12^i for all $i \geq 4$ by Theorem 1.4.

There exists a 4-RGDD of type 5^4 by Theorem 1.4. This is also a $\{4, 5\}$-URGDD of type 4^5 $r_4 = 4$, $r_5 = 1$, which we take as the master design. We take a 3-RGDD of type 3^5 and a $\{3, 4\}$-URGDD of type 3^4 with $r_4 = 1$ from Lemma 2.4 as ingredient designs. We expand all points of the master design three times and obtain a $\{3, 4\}$-URGDD of type 12^5 with $r_4 = 4$.

There exists a 4-RGDD of type 4^{10} with $r_4 = 12$ by Theorem 1.4. We take a $\{3, 4\}$-URGDD of type 3^4 with $r_4 = 1$ from Lemma 2.4 as ingredient design. We expand all points of the master design three times and obtain a $\{3, 4\}$-URGDD of type 12^{10} with $r_4 = 12$.

There exists a 4-RGDD of type 4^{13} by Theorem 1.4, which we take as master design. We take the RGDDs of Lemma 2.5 as ingredient designs. We expand all points of the master design 15 times and obtain a $\{3, 4\}$-URGDD of type 60^{13} with $r_4 \in \{16, 18, \ldots, 240\}$. There exists a $\{3, 4\}$-URGDD of type 12^5 with $r_4 \in \{0, 4, 16\}$ from above. We fill all groups of size 60 with the same URGDD of type 12^5. We obtain a $\{3, 4\}$-URGDD of type 12^{65} with $r_4 \in \{16, 18, \ldots, 256\}$. There exists a 4-RGDD of type $(12 \cdot 65)^5$ with $r_4^0 = 1,040$ by Theorem 1.4. We fill all groups with the same URGDD of type 12^{65}. The result is a $\{3, 4\}$-URGDD of type 12^{325} with $r_4 \in \{1{,}056, 1{,}058, \ldots, 1{,}296\}$. □

Lemma 4.16. *There exists a* $\{3, 4\}$-*URGDD of type* 60^6, $r_4 \in \{0, 2, \ldots, 90, 100\}$.
There exists a $\{3, 4\}$-URGDD *of type* 60^7, $r_4 \in \{0, 4, 6, 8, \ldots, 120\}$.
There exists a $\{3, 4\}$-URGDD *of type* 60^{24}, $r_4 \in \{0, 2, 4, \ldots, 460\}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 4^6 with $r_4 \in \{0, 2, 4, 6\}$ by Lemma 3.1, which we take as master design. We take the RGDDs of Lemma 2.5 as ingredient designs. We expand all points of the master design 15 times. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 1, 3, …, or 15 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 60^6 with $r_4 \in \{0, 2, \ldots, 90, 100\}$.

There exists a 5-RGDD of type 7^5 by Theorem 1.3. This is equivalent to a $\{5, 7\}$-URGDD of type 5^7 $r_5 = 6, r_7 = 1$,which is our master design. We take a 3-RGDD of type 12^7 (Theorem 1.4) and a $\{3, 4\}$-URGDD of type 12^5 with $r_4 \in \{0, 4, 12\}$ (Lemma 4.15) as ingredient designs. We expand all points of the master design 12 times and obtain a $\{3, 4\}$-URGDD of type 60^7 with $r_4 = 4$.

There exists a 4-RGDD of type 7^4 by Theorem 1.4. This is also a $\{4, 7\}$-URGDD of type 4^7 $r_4 = 6, r_7 = 1$,which is our master design. We take a 3-RGDD of type 15^7 (Theorem 1.4) and a $\{3, 4\}$-URGDD of type 15^4 with $r_4 \in \{1, 3, 5, 7, 9, 11, 13, 15\}$ (Lemma 2.5) as ingredient designs. We expand all points of the master design 15 times and obtain a $\{3, 4\}$-URGDD of type 60^7 with $r_4 = 6$. The assertion follows for $u = 7$ by Lemma 4.14.

There exists a 4-RGDD of type 240^6 with $r_4 = 400$ by Theorem 1.4. There exists a $\{3, 4\}$-URGDD of type 4^6 with $r_4 \in \{0, 2, 4, 6\}$ by Lemma 3.1, which we take as master design. We take the URGDDs of Lemma 2.10 as ingredient designs and expand all points of the master design 60 times. We thus obtain a $\{3, 4\}$-URGDD of type 240^6 with $r_4 \in \{0, 2, 4, \ldots, 360, 400\}$. We fill in all groups with the same $\{3, 4\}$-URGDD of type 60^4 with $r_4 \in \{0, 2, 4, \ldots, 60\}$ and get a $\{3, 4\}$-URGDD of type 60^{24} with $r_4 \in \{0, 2, 4, \ldots, 460\}$. □

Lemma 4.17. *There exists a* $\{3,4\}$*-frame of type* 180^{2i+1} *for* $i \geq 2$*,* $i \notin \{3, 11, 13, 19, 23\}$ *and* $\tilde{r}_4 \in \{0, 2, 4, \ldots, 40\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 30^{2i+1} for $i \geq 2$, $i \notin \{3, 11, 13, 19, 23\}$ with $\widehat{r}_4 = 10$ per group by Theorem 1.6, which we take as the master design. We take the RGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a $\{3, 4\}$-frame of type 180^{2i+1} with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 40\}$ per group of the frame. □

Lemma 4.18. *There exists a* $\{3,4\}$*-URGDD of type* 60^{6i+4} *for* $i \geq 1$ *and* $r_4 \in \{0, 2, 4, \ldots, 40(2i+1)\}$.

Proof. There exists a $\{3,4\}$-frame of type 180^{2i+1} for $i \geq 2$ and $\tilde{r}_4 \in \{0, 2, 4, \ldots, 40\}$ per group of the frame by Lemma 4.17. There exists a $\{3,4\}$-URGDD of type 60^4 with $r_4 \in \{0, 2, 4, \ldots, 60\}$ by Lemma 2.10. Adjoin 60 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. Then the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. The result is a $\{3,4\}$-URGDD of type 60^{6i+4} for $i \geq 2$ and $r_4 \in \{0, 2, 4, \ldots, 40(2i+1)\}$.

Now the case $i = 1$. There exists a 4-RGDD of type 10^{10} with $r_4^0 = 30$ by Theorem 1.4, which we take as the master design. We take the URGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. We obtain a $\{3, 4\}$-URGDD of type 60^{10} with $r_4 \in \{0, 2, 4, \ldots, 120\}$, as we fill all parallel classes appropriately. □

Theorem 4.19. *There exists a* $\{3,4\}$*-URGDD of type* 60^{6i+4} *with* $r_4 \in \{0, 2, 4, \ldots, 60(2i+1)\}$ *for* $i \geq 1$.

Proof. The assertion follows by Lemmas 4.14 and 4.18, since $6i + 4 = 3(2i + 1) + 1$. □

Lemma 4.20. *There exists a* $\{3,4\}$*-URGDD of type* 60^{6u+1} *for* $u \geq 5$ *and* $r_4 \in \{0, 4, 6, 8, \ldots, 120u\}$.

Proof. There exists a $\{3,4\}$-frame of type 360^u for $u \geq 5$ and $\tilde{r}_4 \in \{0, 2, 4, \ldots, 80\}$ per group of the frame by Lemma 4.7. There exists a $\{3,4\}$-URGDD of type 60^7 with $r_4 \in \{0, 4, 6, 8, \ldots, 120\}$ by Lemma 4.16.

Adjoin 60 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. Then the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. The result is a $\{3,4\}$-URGDD of type 60^{6u+1} for $u \geq 5$ and $r_4 \in \{0, 4, 6, 8, \ldots, 80u\}$.

There exists a $\{3, 4\}$-frame of type 360^u for $u \geq 5$ and $\tilde{r}_4 \in \{8, 10, 12, \ldots, 120\}$ per group of the frame by Lemma 4.8. Adjoin 60 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. The result is a $\{3, 4\}$-URGDD of type 60^{6u+1} for $u \geq 5$ and $r_4 \in \{8u, 8u + 2, 8u + 4, \ldots, 120u\}$.

Remark, that it is no simple way to combine both frames, while for example we have no frame with $\tilde{r}_4 < 8$ in one group and $\tilde{r}_4 > 80$ in another group. □

Lemma 4.21. *There exists a* $\{3, 4\}$*-URGDD of type* 72^u *with* $r_4 \in \{0, 2, 4, \ldots, 16(u - 1), 24(u - 1)\}$ *for* $u \geq 4$.

Proof. There exists a 4-RGDD of type 72^u with $r_4^0 = 24(u - 1)$ for $u \geq 4$ by Theorem 1.4. There exists a 4-RGDD of type 12^u with $r_4^0 = 4(u - 1)$ for $u \geq 4$ by Theorem 1.4, which we take as the master design. We take the URGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 72^u with $r_4 \in \{0, 2, 4, \ldots, 16(u - 1)\}$, as we fill all parallel classes appropriately. □

Lemma 4.22. *There exists a* $\{3, 4\}$*-URGDD of type* 72^{3i+1} *with* $r_4 \in \{0, 2, 4, \ldots, 72i\}$ *for* $i \geq 1$.

Proof. There exists a 4-RGDD of type 8^{3i+1} with $r_4^0 = 8i$ for $i \geq 1$ by Theorem 1.4, which we take as the master design. We take the RGDDs of Lemma 2.2 as ingredient designs. We expand all points of the master design nine times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 1, 3, 5, 7, or 9 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 72^u with $r_4 \in \{8i, 8i + 2, 8i + 4, \ldots, 72i\}$ for $i \geq 1$, as we fill all parallel classes appropriately. The assertion follows by Lemma 4.21. □

Lemma 4.23. *There exists a* $\{3, 4\}$*-URGDD of type* 84^{3i+1} *for* $i \geq 1$ *and* $r_4 \in \{0, 4i, 4i + 2, \ldots, 84i\}$.

Proof. There exists a 3-RGDD of type 84^{3i+1} for $i \geq 1$ by Theorem 1.4.

There exists a 4-RGDD of type 4^{3i+1} with $r_4^0 = 4i$ for $i \geq 1$ by Theorem 1.4, which we take as the master design and all designs of Lemma 2.7 as ingredient designs. We expand all points of the master design 21 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in $1, 3, \ldots, 19$, or 21 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 84^{3i+1} with $r_4 \in \{4i, 4i + 2, \ldots, 84i\}$, as we fill all parallel classes appropriately. □

Lemma 4.24. *There exists a* $\{3, 4\}$*-URGDD of type* 216^u *for* $u \geq 4$ *and* $r_4 \in \{0, 2, 4, \ldots, 72(u - 1)\}$.

Proof. There exists a uniformly resolvable {3,4}-URGDD of type 18^4, $r_4 \in \{0, 2, 4, \ldots, 18\}$ by Lemma 2.6, which is our ingredient design. There exists a 4-RGDD of type 12^u for $u \geq 4$ by Theorem 1.4, which we take as the master design. We expand all points of the master design 18 times. We obtain a {3, 4}-URGDD of type 216^u with $r_4 \in \{0, 2, 4, \ldots, 72(u-1)\}$, as we fill all parallel classes appropriately. □

Lemma 4.25. *There exists a* $\{3, 4\}$*-URGDD of type* 12^{12i+4} *for* $i \geq 1$ *and* $r_4 \in \{0, 2, 4, \ldots, 48i + 12\}$.

Proof. There exists a 4-RGDD of type 1^{12i+4} with $r_4^0 = 4i + 1$ for $i \geq 1$ by Theorem 1.4, which we take as the master design. We take the RGDDs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 0, 2, 4, 6, 8, 10, or 12 4-pcs. We obtain a {3, 4}-URGDD of type 12^{12i+4} with $r_4 \in \{0, 2, 4, \ldots, 48i + 12\}$ for $i \geq 1$, as we fill all parallel classes appropriately. □

Lemma 4.26. *There exists a* $\{3, 4\}$*-URGDD of type* 12^{6i+4} *for* $i \geq 2$, $i \notin \{13, 19\}$ *and* $r_4 \in \{0, 2, 4, \ldots, 8(2i + 1), 12(2i + 1)\}$.

Proof. There exists a {3, 4}-frame of type 36^{2i+1} for $i \geq 2$, $i \notin \{3, 11, 13, 17, 19, 23\}$ and $\tilde{r}_4 \in \{0, 2, 4, 6, 8\}$ per group of the frame by Lemma 4.5. There exists a {3, 4}-URGDD of type 12^4 with $r_4 \in \{0, 2, \ldots, 12\}$ by Lemma 2.3. Adjoin 12 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. Then the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. We obtain a {3, 4}-URGDD of type 12^{6i+4} for $i \geq 2$, $i \notin \{3, 11, 13, 17, 19, 23\}$ and $r_4 \in \{0, 2, 4, \ldots, 8(2i + 1)\}$.

There exists a 4-RGDD of type 2^{6i+4} for $i \in \{3, 17, 23\}$ by Theorem 1.4, which we take as master design and all designs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. We obtain a {3, 4}-URGDD of type 12^{6i+4} with $r_4 \in \{0, 2, 4, \ldots, 8(2i + 1)\}$ for $i \in \{3, 17, 23\}$, as we fill all parallel classes appropriately.

There exists a {3, 4}-URGDD of type 120^7 with $r_4 \in \{0, 2, 4, \ldots, 160\}$ by Lemma 4.11. Filling all groups with a 3-RGDD of type 12^{10} or a 4-RGDD of type 12^{10} as appropriate results in all {3, 4}-URGDD of type 12^{70} with $r_4 \in \{0, 2, 4, \ldots, 160 + 36\}$. □

Lemma 4.27. *There exists a* $\{3, 4\}$*-URGDD of type* 12^{15u+1} *for* $u \geq 5$ *and* $r_4 \in \{4u, 4u + 2, 4u + 4, \ldots, 60u\}$.

Proof. There exists a {3, 4}-frame of type 180^u for $u \geq 5$ and $\tilde{r}_4 \in \{4, 6, 8, \ldots, 60\}$ per group of the frame by Lemma 3.21. There exists a {3, 4}-URGDD of type 12^{15+1} with $r_4 \in \{0, 2, 4, \ldots, 60\}$ by Lemma 4.25. Adjoin 12 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. Then the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. The result is a $\{3, 4\}$-URGDD of type 12^{15u+1} for $u \geq 5$ and $r_4 \in \{4u, 4u + 2, 4u + 4, \ldots, 60u\}$. □

Lemma 4.28. *There exists a* $\{3, 4\}$*-URGDD of type* 12^{30i+16} *for* $i \geq 2$ *and* $r_4 \in \{0, 2, 4, \ldots, 60(2i + 1)\}$.

Proof. Let $j = 5i + 2$. We have $30i + 16 = 6(5i + 2) + 4 = 6j + 4$. For $i \in \{3, 11, 13, 19, 23\}$, $j \in \{17, 57, 67, 97, 117\}$, respectively, there exists a $\{3, 4\}$-URGDD of type 12^{6j+4} with $r_4 \in \{0, 2, 4, \ldots, 8(2j + 1)\}$ by Lemma 4.26.

There exists a $\{3, 4\}$-frame of type 180^{2i+1} for $i \geq 2$, $i \notin \{3, 11, 13, 19, 23\}$ and $\tilde{r}_4 \in \{0, 2, 4, \ldots, 40\}$ per group of the frame by Lemma 4.17 . There exists a $\{3, 4\}$-URGDD of type 12^{15+1} with $r_4 \in \{0, 2, 4, \ldots, 60\}$ by Lemma 4.25. Adjoin 12 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. Then the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. The result is a $\{3, 4\}$-URGDD of type 12^{30i+16} for $i \geq 2$ and $r_4 \in \{0, 2, 4, \ldots, 40(2i + 1)\}$.

The assertion follows in the same way by use of Lemma 3.21 with $u = 2i + 1$. □

Lemma 4.29. *There exists a* $\{3, 4\}$*-URGDD of type* 24^{2i+1} *with* $r_4 \in \{0, 2, 4, \ldots, 8i\}$ *for* $i \geq 2$.

Proof. Let $i \geq 2$. There exists a 4-RGDD of type $(2i + 1)^4$ by Theorem 1.4. This is also a $\{4, 2i + 1\}$-URGDD of type $4^{2i+1} r_4 = 2i$, $r_{2i+1} = 1$, which we take as the master design. We take the RGDDs of Lemma 2.1 as ingredient designs. We expand all points of the master design six times. All blocks of any parallel class have to be filled with the same ingredient design. Each 4-pc of the master design results in 0, 2, or 4 4-pcs. There exists a 3-RGDD of type 6^{2i+1} by Theorem 1.4. We obtain a {3,4}-URGDD of type 24^{2i+1} with $r_4 \in \{0, 2, 4, \ldots, 8i\}$, as we fill all parallel classes appropriately. □

Lemma 4.30. *There exists a* $\{3, 4\}$*-URGDD of type* 24^{6u+1} *for* $u \geq 5$ *and* $r_4 \in \{0, 2, 4, \ldots, 48u - 4, 48u\}$.

Proof. We take a $\{3, 4\}$-frame of type 144^u for $u \geq 5$ and $\tilde{r}_4 \in \{0, 2, 4, \ldots, 48\}$ per group of the frame by Lemma 4.3. There exists a $\{3, 4\}$-URGDD of type 24^7 with $r_4 \in \{0, 2, 4, \ldots, 44, 48\}$ by Lemma 3.15. Adjoin 24 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. Then the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. □

Lemma 4.31. *There exists a* $\{3, 4\}$*-URGDD of type* 108^u *for* $u \geq 4$ *and* $r_4 \in \{0, 4(u - 1), 4(u - 1) + 2, \ldots, 36(u - 1)\}$.

Proof. There exists a 3-RGDD of type 108^u for $u \geq 4$ by Theorem 1.4.

There exists a 4-RGDD of type 12^u for $u \geq 4$ by Theorem 1.4, which we take as the master design and all designs of Lemma 2.2 as ingredient designs. We expand all points of the master design nine times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each parallel class expands in a way that several uniform parallel classes are created. Each 4-pc of the master design results in 1, 3, 5, 7, or 9 4-pcs. We obtain a {3, 4}-URGDD of type 108^u with $r_4 \in \{0, 4(u-1), 4(u-1)+2, \ldots, 36(u-1)\}$, as we fill all parallel classes appropriately. □

Lemma 4.32. *There exists a* $\{3, 4\}$-*frame of type* 324^u *for* $u \geq 5$ *and* $\tilde{r}_4 \in \{4, 6, 8, \ldots, 36\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 12^u for $u \geq 5$ with $\widehat{r}_4 = 4$ per group by Theorem 1.6, which we take as the master design. We take the URGDDs of Lemma 2.8 as ingredient designs. We expand all points of the master design 27 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 1, 3, 5, 7, or 9 4-pcs. We obtain a {3, 4}-frame of type 324^u with $\tilde{r}_4 \in \{4, 6, 8, \ldots, 36\}$ per group of the frame. □

Lemma 4.33. *There exists a* $\{3, 4\}$-*frame of type* 324^u *for* $u \geq 5$, $u \neq 12$, *and* $\tilde{r}_4 \in \{12, 14, \ldots, 108\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 36^u with $\widehat{r}_4 = 12$ per group for $u \geq 5, u \neq 12$ by Theorem 1.6, which we take as the master design. We take the URGDDs of Lemma 2.2 as ingredient designs. We expand all points of the master design nine times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 1, 3, 5, 7, or 9 4-pcs. We obtain a {3, 4}-frame of type 324^u with $\tilde{r}_4 \in \{12, 14, \ldots, 108\}$ per group of the frame. □

Lemma 4.34. *There exists a* $\{3, 4\}$-URGDD *of type* 12^{27u+1} *for* $u \geq 5$, $u \neq 12$ *and* $r_4 \in \{4u, 4u+2, \ldots, 108u\}$.

Proof. There exists a {3, 4}-frame of type 324^u for $u \geq 5$ and $\tilde{r}_4 \in \{4, 6, 8, \ldots, 36\}$ per group of the frame by Lemma 4.32. There exists a {3, 4}-URGDD of type 12^{27+1} with $r_4 \in \{0, 2, 4, \ldots, 108\}$ by Lemma 4.25. Adjoin 12 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. The result is a {3, 4}-URGDD of type 12^{27u+1} for $u \geq 5$ and $r_4 \in \{4u, 4u+2, \ldots, 36u\}$.

There exists a {3, 4}-frame of type 324^u for $u \geq 5, u \neq 12$, and $\tilde{r}_4 \in \{12, 14, \ldots, 108\}$ per group of the frame by Lemma 4.33. Adjoin 12 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. The result is a {3, 4}-URGDD of type 12^{27u+1} for $u \geq 5$, $u \neq 12$, and $r_4 \in \{12u, 12u+2, \ldots, 108u\}$. □

Lemma 4.35. *There exists a* $\{3, 4\}$-*URGDD of type* 36^{4u+1} *for* $u \geq 5$ *and* $r_4 \in \{0, 4, 6, 8, \ldots, 48u - 12, 48u\}$.

Proof. We take a {3, 4}-frame of type 144^u for $u \geq 5$ and $\tilde{r}_3 \in \{72, 69, 66, \ldots, 0\}, \tilde{r}_4 \in \{0, 2, 4, \ldots, 48\}$ per group of the frame from Lemma 4.3. There exists a {3, 4}-URGDD of type 36^5 with $r_4 \in \{0, 4, 6, 8, \ldots, 36, 48\}$ by Lemma 3.12 and Theorem 1.4. Adjoin 36 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. Then, there are an equal number of 3-pcs corresponding to the group of the frame and its URGDD. The result is a {3, 4}-URGDD of type 36^{4u+1} for $u \geq 5$ and $r_4 \in \{0, 4, 6, 8, \ldots, 48u - 12, 48u\}$. □

Lemma 4.36. *There exists a* $\{3, 4\}$*-URGDD of type* 24^u *with* $r_4 \in \{0, 2, 4, \ldots, 7(u-1)\}$ *for* $u \in \{11, 17, 23, 41, 59\}$.

Proof. There exists an RTD$(8, u)$ for $u \in \{11, 23, 41, 59\}$, since all these u are prime. Therefore, there exists a $\{8, u\}$-URGDD of type 8^u with $r_8 = u - 1$ and $r_u = 1$. We apply the latter as the master design. There exist a {3, 4}-URGDD of type 3^8, $r_4 \in \{1, 3, 5, 7\}$ by Lemma 3.1 and a 3-RGDD of type 3^u by Theorem 1.4, which we take as ingredient designs. We expand all points of the master design three times. All blocks of any parallel class have to be filled with the same ingredient design. Each 8-pc of the master design results in 1, 3, 5, or 7 4-pcs. We obtain a {3, 4}-URGDD of type 24^u with $r_4 \in \{u - 1,\ u + 1, \ldots, 7(u - 1)\}$, as we fill all parallel classes appropriately. The assertion follows by Lemma 4.29. □

Lemma 4.37. *There exists a* $\{3, 4\}$*-frame of type* 108^{4i+1} *for* $i \geq 1$ *and* $\tilde{r}_4 \in \{0, 2, 4, \ldots, 36\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 9^{4i+1} for $i \geq 1$ with $\widehat{r}_4 = 3$ per group by Theorem 1.6, which we take as the master design. We take the URGDDs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 0, 2, 4 . . ., 12 4-pcs. We obtain a {3, 4}-frame of type 108^{4i+1} with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 36\}$ per group of the frame. □

Lemma 4.38. *There exists a* $\{3, 4\}$*-frame of type* 252^{4i+1} *for* $i \geq 1$ *and* $\tilde{r}_4 \in \{0, 2, 4, \ldots, 84\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 21^{4i+1} for $i \geq 1$ with $\widehat{r}_4 = 7$ per group by Theorem 1.6, which we take as the master design. We take the URGDDs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 0, 2, 4, . . . , 12 4-pcs. We obtain a {3, 4}-frame of type 252^{4i+1} with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 84\}$ per group of the frame. □

Lemma 4.39. *There exists a* $\{3, 4\}$-frame *of type* $1,008^u$ *for* $u \geq 5$ *and* $\tilde{r}_4 \in \{0, 2, 4, \ldots, 336\}$ *per group of the frame. This* $\tilde{r}_4$ *can be chosen independently for each group.*

Proof. There exists a 4-frame of type 84^u for $u \geq 5$ with $\widehat{r}_4 = 28$ per group by Theorem 1.6, which we take as the master design. We take the RGDDs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times. All blocks of any parallel class have to be filled with the same ingredient design. Therefore, each holey parallel class expands in a way that several uniform holey parallel classes are created. Each 4-pc of the master design results in 0, 2, 4, 6, 8, 10, or 12 4-pcs. We obtain a $\{3, 4\}$-frame of type $1,008^u$ with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 336\}$ per group of the frame. □

Lemma 4.40. *There exists an* $IURD(\{3, 4\}; 1,008 + 264)$ *with a hole of size 264 and* $r_4 \in \{0, 2, 4, \ldots, 308, 336\}$, $r_4^0 \in \{1, 3, 5, \ldots, 85\}$.

Proof. There exists a $\{3, 4\}$-frame of type 252^5 and $\tilde{r}_4 \in \{0, 2, 4, \ldots, 84\}$ per group of the frame by Lemma 4.38. There exists a $\{3, 4\}$-URGDD of type 12^{21+1} with $r_4 \in \{0, 2, 4, \ldots, 56, 84\}$ by Lemma 4.26. Adjoin 12 infinite points to the frame and fill four groups with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group. Then the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. The result are 4-pcs with $r_4 \in \{0, 2, 4, \ldots, 308, 336\}$. We fill in the 24 groups of size 12, which are enclosed in the chosen four groups of size 72, with a URD($\{3, 4\}$; 12) and obtain four partial 3-pcs and one partial 4-pc. Together with the partial 4-pcs of the last group, we have $r_4^0 \in \{1, 3, 5, \ldots, 85\}$ partial 4-pcs. The last group and the infinite points generate the hole of size 264. □

Lemma 4.41. *There exists a* $\{3, 4\}$*-URGDD of type* 48^{3i+1} *for* $i \geq 1$ *and* $r_4 \in \{0, 2, 4, \ldots, 48i\}$.

Proof. There exists a 4-RGDD of type 4^{3i+1} for $i \geq 1$ by Theorem 1.4, which we take as the master design and all designs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times. We obtain a $\{3, 4\}$-URGDD of type 48^{3i+1} with $r_4 \in \{0, 2, 4, \ldots, 48i\}$, as we fill all parallel classes appropriately. □

Lemma 4.42. *There exists a* $\{3, 4\}$*-URGDD of type* 48^6 *with* $r_4 \in \{0, 2, 4, \ldots, 72, 80\}$. *There exists a* $\{3, 4\}$*-URGDD of type* 48^{11} *with* $r_4 \in \{0, 2, 4, \ldots, 160\}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 4^6 with $r_4 \in \{0, 2, 4, 6\}$ by Lemma 3.1, which we take as master design. We take the RGDDs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times. We obtain a $\{3, 4\}$-URGDD of type 48^6 with $r_4 \in \{0, 2, 4, \ldots, 72, 80\}$.

There exists a 4-RGDD of type 11^4 by Theorem 1.4. This is also a $\{4, 11\}$-URGDD of type 4^{11} $r_4 = 10$, $r_{11} = 1$, which we take as master design. We take a 3-RGDD of type 12^{11}, a 4-RGDD of type 12^{11}, and all designs of Lemma 2.3 as ingredient designs. We expand all points of the master design 12 times. We obtain a $\{3, 4\}$-URGDD of type 48^{11} with $r_4 \in \{0, 2, 4, \ldots, 160\}$, as we fill all parallel classes appropriately. □

Lemma 4.43. *There exists a* $\{4, 6\}$*-frame of type* $(3; 4^1)^{2(2i-1)}(5; 6^1)^1$ *for* $i \geq 4$ *and* $i \neq 34$.

Proof. There exists a 4-RGDD of type 6^{2i} for $i \geq 4$ and $i \neq 34$ by Theorem 1.4. We remove a point and obtain a $\{4, 6\}$-frame of type $(3; 4^1)^{2(2i-1)}(5; 6^1)^1$. □

Lemma 4.44. *There exists a {6, g}-frame of type* $(5; 6^1)^g(g-1; g^1)^1$ *for* $g \geq 7$ *and* $g \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$.

Proof. There exists an RTD$(6, g)$ for $g \geq 7$ and $g \notin \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$ by Theorem 1.3. Therefore, there exists a $\{6, g\}$-*URGDD* of type 1^{6g} with $r_6 = g$ and $r_g = 1$. We remove a point and obtain a $\{6, g\}$-frame of type $(5; 6^1)^g(g-1; g^1)^1$ for the same g. □

5. NEW CLASSES OF URDS

In this section, we derive the existence of URDs for some new modular classes. Specifically, we show that all admissible URDs exist for $v \equiv 36 \pmod{144}$, $v \equiv 0 \pmod{60}$, and $v \equiv 36 \pmod{108}$, with a few possible exceptions. Our first main result follows.

Theorem 5.1. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 36 \pmod{144}$, *possibly excepting* $v = 612$: $r_4 \in \{189, 191\}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 36^{4u+1} for $u \geq 5$ and $r_4 \in \{0, 4, 6, 8, \ldots, 48u-12, 48u\}$ by Lemma 4.35. Filling in all groups with the same appropriate URD$(\{3, 4\}; 36)$ (Lemma 3.2) results in all admissible URD$(\{3, 4\}; 144u + 36)$ for $u \geq 5$, while the gaps are covered by URD$(\{3, 4\}; 36)$ with $r_4 \in \{1, 3, 5, \ldots, 11\}$.

There exist all admissible URD$(\{3, 4\}; 180)$ by Lemma 3.17.

There exists a $\{3, 4\}$-URGDD of type 4^9 with $r_4 \in \{0, 2, 4, 6, 8, 10\}$ by Lemma 3.2. We expand all points of this design nine times and obtain a $\{3, 4\}$-URGDD of type 36^9 with $r_4 \in \{0, 2, \ldots, 90, 96\}$ by Lemma 2.2 and Theorem 1.4. By filling all groups appropriately with the same URD$(\{3, 4\}; 36)$, we obtain all admissible URD$(\{3, 4\}; 324)$.

There exist all admissible URD$(\{3, 4\}; 468)$ by Lemma 3.26.

There exists a $\{3, 4\}$-URGDD of type 36^{17} with $r_4 \in \{0, 2, 4, \ldots, 176, 192\}$ by Lemma 3.24. We obtain all admissible URD$(\{3, 4\}; 612)$ possibly excepting $r_4 \in \{189, 191\}$ by filling all groups appropriately with the same URD$(\{3, 4\}; 36)$. □

Lemma 5.2. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 0 \pmod{360}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 36^{10} for $r_4 \in \{0, 12, 14, 16, \ldots, 108\}$ by Lemma 3.13. We obtain all admissible URD$(\{3, 4\}; 360)$ by filling all groups appropriately with the same URD$(\{3, 4\}; 36)$.

There exists a $\{3, 4\}$-URGDD of type 180^{2i} for $i \geq 2$ and $r_4 \in \{0, 2, 4, \ldots, 60(2i-1)\}$ by Theorem 4.10. There exist all admissible URD$(\{3, 4\}; 180)$ by Lemma 3.17. The assertion follows by filling all groups appropriately with the same URD$(\{3, 4\}; 180)$. □

Lemma 5.3. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 120 \pmod{360}$, *possibly excepting* $v = 120$: $r_4 \in \{27, 29, 31\}$.

Proof. The case $v = 120$ is handled in Lemma 3.9.

There exists a $\{3, 4\}$-URGDD of type 120^{3i+1} for $i \geq 1$ and $r_4 \in \{0, 2, 4, \ldots, 120i\}$ by Theorem 4.13. There exist all admissible URD$(\{3, 4\}; 120)$ possibly excepting

$r_4 \in \{27, 29, 31\}$ by Lemma 3.9. There exists a URD($\{3, 4\}; 360i + 120$), $r_4 \in \{1, 3, 5, \ldots, 120i + 23\}$ by filling all groups appropriately with the same URD($\{3, 4\}; 120$).

There exists a 4-RGDD of type 60^{6i+2} with $r_4^0 = 20(6i + 1)$ for $i \geq 1$ by Theorem 1.4. We obtain a URD($\{3, 4\}; 360i + 120$), $r_4 \in \{120i + 21, 120i + 23, \ldots, 120i + 39\}$ by filling all groups appropriately with the same URD($\{3, 4\}; 60$) (Lemma 3.4). □

Lemma 5.4. *There exist all admissible URD*($\{3, 4\};\ v$) *for* $v \equiv 240 \pmod{360}$.

Proof. There exists a URD($\{3, 4\}; 240$) by Theorem 1.14.

There exists a $\{3, 4\}$-URGDD of type 60^{6i+4} with $r_4 \in \{0, 4, 6, 8, \ldots, 60(2i + 1)\}$ for $i \geq 1$ by Theorem 4.19. There exist all admissible URD($\{3, 4\}; 60$) by Lemma 3.4. The assertion follows by filling all groups appropriately with the same URD($\{3, 4\}$; 60). □

Theorem 5.5. *There exist all admissible URD*($\{3, 4\};\ v$) *for* $v \equiv 0 \pmod{120}$*, possibly excepting* $v = 120$*:* $r_4 \in \{27, 29, 31\}$.

Proof. The assertion follows by Lemmas 5.2–5.4. □

Lemma 5.6. *There exists a* $\{3, 4\}$*-URGDD of type* 60^{2i+1} *with* $r_4 \in \{0, 4, 8, \ldots, 40i - 24,\ 40i - 16, 40i - 12,\ 40i\}$ *for* $i \geq 2$.

Proof. There exists a 5-RGDD of type $(2i + 1)^5$ for $i \geq 2$ by Theorem 1.3. This is also a $\{5, 2i+1\}$-URGDD of type 5^{2i+1}, $r_5^0 = 2i$, $r_{2i+1}^0 = 1$, which we take as the master design. There exist a 3-RGDD of type 12^{2i+1}, a 4-RGDD of type 12^{2i+1} with $r_4^0 = 8i$, and a $\{3, 4\}$-URGDD of type 12^5 with $r_4 \in \{0, 4, 16\}$ by Lemma 4.15, which we take as ingredient designs. We expand all points of the master design 12 times. Each 5-pc of the master design results in 0, 4, or 16 4-pcs. We obtain a $\{3, 4\}$-URGDD of type 60^{2i+1} with $r_4 \in \{0, 4, 8, \ldots, 32i - 24, 32i - 16, 32i - 12, 32i\} \cup \{8i, 8i + 4, 8i + 8, \ldots, 40i - 24, 40i - 16, 40i - 12, 40i\}$, as we fill all parallel classes appropriately. □

Lemma 5.7. *There exist all admissible URD*($\{3, 4\}; 120i + 60$) *for* $i \geq 1$.

Proof. There exist all admissible URD($\{3, 4\}; 180$) by Lemma 3.17. There exists a $\{3, 4\}$-URGDD of type 60^{2i+1} with $r_4 \in \{0, 4, 8, \ldots, 40i - 24,\ 40i - 16, 40i - 12,\ 40i\}$ for $i \geq 2$ by Lemma 5.6. By filling with URD($\{3, 4\}; 60$) (Lemma 3.4), we obtain a URD($\{3, 4\}; 120i + 60$) with $r_4 \in \{1, 3, \ldots, 40i + 19\}$ for $i \geq 2$. □

Now we are ready for our second main result.

Theorem 5.8. *There exist all admissible URD*($\{3, 4\};\ v$) *for* $v \equiv 0 \pmod{60}$*, possibly excepting* $v = 120$*:* $r_4 \in \{27, 29, 31\}$.

Proof. The assertion follows by Lemma 3.4, Theorem 5.5, and Lemma 5.7. □

Theorem 5.9. *There exist all admissible URD*($\{3, 4\};\ v$) *for* $v \equiv 72 \pmod{216}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 72^{3i+1} with $r_4 \in \{0, 2, 4, \ldots, 72i\}$ for $i \geq 1$ by Lemma 4.22. There exist all admissible URD($\{3, 4\}; 72$) by Lemma 3.5. The assertion follows by filling all groups appropriately with the same URD($\{3, 4\}; 72$). □

Lemma 5.10. *There exist all admissible URD*$(\{3,4\}; 108\,u+36)$ *for* $u \geq 5$*, possibly excepting* $r_4 \in \{11,\ 13, \ldots, 4u-1\}$ *for* $u \equiv 0 \pmod 2$.

Proof. There exists a $\{3,4\}$-frame of type 108^u with $\tilde{r}_4 \in \{4, 6, \ldots, 36\}$ 4-pcs per group for all $u \geq 5$ from Lemma 4.2. There exists a $\{3,4\}$-URGDD of type 36^4 with $r_4 \in \{0, 4, 6, 8, \ldots, 36\}$ by Lemma 3.13. Adjoin 36 infinite points to the frame and fill each group together with the infinite points with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group of the frame. Therefore, the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. The result is a $\{3,4\}$-URGDD of type 36^{3u+1} for $u \geq 5$ and $r_4 \in \{4u, 4u+2, 4u+4, \ldots, 36u\}$. Filling in all groups with the same appropriate URD$(\{3,4\}; 36)$ from Lemma 3.17 results in all admissible URD$(\{3,4\};\ 108u\ +\ 36)$ for $u \geq 5$ and $r_4 \in \{4u+1, 4u+3, 4u+5, \ldots, 36u+11\}$.

When $u \equiv 1 \pmod 2$, $u \geq 5$, and $u \notin \{15,\ 23,\ 27\}$, there exists a $\{3,4\}$-frame of type 108^u with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 24\}$ 4-pcs per group by Lemma 4.1. Proceeding as above, using this frame in place of that above, gives a $\{3,4\}$-URGDD of type $36^{3u+1}$with $r_4 \in \{0, 2, 4, \ldots, 24u\}$. Filling in all groups with the same appropriate URD$(\{3,4\};$ $36)$ from Lemma 3.17 results in all admissible URD$(\{3,4\};\ 108u\ +\ 36)$ with $r_4 \in \{1, 3, 5, \ldots, 24u+11\}$ when $u \geq 5$, $u \equiv 1 \pmod 2$, and $u \notin \{15,\ 23,\ 27\}$.

To deal with the cases $u \in \{15,\ 23,\ 27\}\}$, take $\hat{u} = (3u+1)/2$, so $\hat{u} \in \{23, 35, 41\}$, respectively. There exists a $\{3,4\}$-RGDD of type $72^{\hat{u}}$ with $\hat{r}_4 \in \{0, 2, 4, \ldots, 16(\hat{u}-1)\}$ by Lemma 4.21. Filling in all groups with the same appropriate URD$(\{3,4\}; 72)$ $\bar{r}_4 \in \{1, 3\}$from Theorem 1.12 results in a URD$(\{3,4\}; 72\hat{u})$ with $r_4 \in \{1, 3, 5, \ldots, 16(\hat{u}-1)+3\}$ for $\hat{u} \in \{23, 35, 41\}$.

Note that $72\hat{u} = 108u + 36$ and $16(\hat{u}-1) = 8(3u+1) = 24u+8$, so there exists a URD$(\{3,4\}; 108u+36)$ with $r_4 \in \{1, 3, 5, \ldots, 24u+11\}$ for $u \in \{15,\ 23,\ 27\}$.

There exists a URD$(\{3,4\}; 108u+36)$ with $r_4 \in \{1, 3, 5, 7, 9\}$ for $u \equiv 0 \pmod 2$ by Theorems 1.4, 1.12, and 2.12. □

Lemma 5.11. *There exist all admissible URD*$(\{3,4\}; 216\,u+36)$ *for* $u \geq 3$.

Proof. We take a $\{3,4\}$-frame of type 216^u for $u \geq 5$ and $\tilde{r}_4 \in \{0, 2, 4, \ldots, 72\}$ per group of the frame from Lemma 4.4. There exists a $\{3,4\}$-URGDD of type 36^7 with $r_4 \in \{0, 2, 4, \ldots, 72\}$ by Lemma 3.23. Adjoin 36 infinite points to the frame and fill each group with one of the above URGDDs, where the infinite points form a group. Each group of the frame has to be filled with a URGDD with the same number of 4-pcs as are corresponding to the group of the frame. Therefore, the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. The result is a $\{3,4\}$-URGDD of type 36^{6u+1} for $u \geq 5$, $r_4 \in \{0, 2, 4, \ldots, 72u\}$. Filling in all groups with the same appropriate URD$(\{3,4\}; 36)$ from Lemma 3.2 results in all admissible URD$(\{3,4\}; 216\,u+36)$ for $u \geq 5$.

There exists a $\{3,4\}$-URGDD of type 36^u, $r_4 \in \{0, 2, \ldots, 8(u-1)\}$ for $u \in \{19, 25\}$by Lemma 3.20. Filling in all groups with the same appropriate URD$(\{3,4\}; 36)$ results in all URD$(\{3,4\}; 648)$ and URD$(\{3,4\}; 900)$ for $r_4 \in \{1, 3, 5, \ldots, 8(u-1)+11\}$. Therefore, there exist all admissible URD$(\{3,4\}; 648)$ and URD$(\{3,4\}; 900)$ by Lemma 5.10. □

Now we are ready for our third main result.

Theorem 5.12. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 36 \pmod{108}$.

Proof. There exist all admissible URD$(\{3, 4\}; 144)$ by Theorem 1.14.
The assertion follows by Lemmas 3.25, 5.10, and 5.11. □

6. URDs FOR *v* CONGRUENT 24 MODULO 48

There exist all admissible URDs for $v \equiv 0$ (mod48) by Theorem 1.14. In this section, we deal with the case $v \equiv 24 \pmod{48}$ by considering the cases v congruent 24, 72, and 120 modulo 144. We firstly obtain the lower half of all admissible values of r_4.

Theorem 6.1. *There exists a URD*$(\{3, 4\}; 48i + 24)$ *with* $r_4 \in \{1, 3, \ldots, 8i + 7\}$.

Proof. There exist all admissible URD$(\{3, 4\}; 72)$ by Lemma 3.5.
There exists a $\{3, 4\}$-URGDD of type 24^{2i+1} with $r_4 \in \{0,\ 2, 4 \ldots, 8i\}$ for $i \geq 2$ by Lemma 4.29. Filling in all groups with the same appropriate URD$(\{3, 4\}; 24)$ results in all desired designs. □

Theorem 6.2. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 24 \pmod{144}$ *possibly excepting* $v = 456$ *and* $r_4 \in \{141, 143\}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 24^{6u+1} for $u \geq 5$ and $r_4 \in \{0, 2, 4, \ldots, 48u - 4,\ 48u\}$ by Lemma 4.30. Filling in all groups with the same appropriate URD$(\{3, 4\}; 24)$ results in all admissible URD$(\{3, 4\};\ v)$ for $v \equiv 24 \pmod{144}$, $v \geq 744$.
There exist all admissible URD$(\{3, 4\}; 168)$ by Lemma 3.16.
There exists a $\{3, 4\}$-URGDD of type 24^{13} with $r_4 \in \{0, 2, 4, \ldots, 88, 96\}$ by Lemma 3.15. Filling in all groups with the same appropriate URD$(\{3, 4\}; 24)$ results in all admissible URD$(\{3, 4\}; 312)$.
There exists a $\{3, 4\}$-URGDD of type 24^{19} with $r_4 \in \{0, 2, 4, \ldots, 132, 144\}$ by Lemma 3.15. Filling in all groups with the same appropriate URD$(\{3, 4\}; 24)$ results in all admissible URD$(\{3, 4\}; 456)$, possibly excepting $r_4 \in \{141, 143\}$. There exist all admissible URD$(\{3, 4\}; 600)$ by Theorem 5.5. □

Lemma 6.3. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 72 \pmod{432}$.

Proof. The assertion follows by Theorem 5.9. □

Theorem 6.4. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 0 \pmod{216}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 216^u for $u \geq 4$ and $r_4 \in \{0, 2, 4, \ldots, 16(u - 1), 16(u - 1) + 14, \ldots, 72(u - 1)\}$ by Lemma 4.24.
Filling in all groups with the same appropriate URD$(\{3, 4\}; 216)$ (Lemma 3.19) results in all admissible URD$(\{3, 4\};\ v)$ for $v \equiv 0 \pmod{216}$, $v \geq 1,080$.
There exist all admissible URD$(\{3, 4\}; 216)$ by Lemma 3.19.
There exist all admissible URD$(\{3, 4\};\ v)$, $v \in \{432, 864\}$ by Theorem 1.14.
There exist a $\{3, 4\}$-URGDD of type 108^6 with $r_4 \in \{0,\ 20, 22, \ldots, 180\}$ by Lemma 4.31. There exist all admissible URD$(\{3, 4\}; 648)$ by filling all groups

appropriately with the same URD($\{3,4\}$; 108) (Lemma 3.7). For example $r_4 = 180 + 29 = 176 + 33$ and $r_4 = 180 + 31 = 178 + 33$. □

Corollary 6.5. *There exist all admissible URD*($\{3,4\};\ v$) *for* $v \equiv 216 \pmod{432}$.

Proof. The assertion follows by Theorem 6.4. □

Lemma 6.6. *There exist all admissible URD*($\{3,4\};\ v$) *for* $v \equiv 360 \pmod{432}$.

Proof. We have $432i + 360 = 108(4i) + 324 + 36 = 108(4i + 3) + 36$. The assertion follows by Theorem 5.12. □

Theorem 6.7. *There exist all admissible URD*($\{3,4\};\ v$) *for* $v \equiv 72 \pmod{144}$.

Proof. The assertion follows by Lemma 6.3, Corollary 6.5, and Lemma 6.6. □

This leaves the case $v \equiv 120 \pmod{144}$. We begin by giving some lemmas and a theorem that we will use later.

Lemma 6.8. *There exist all admissible URD*($\{3,4\}; 324u + 12$) *for* $u \geq 5$, *possibly excepting* $\hat{r}_4 \in \{11, 13, \ldots, 4u - 1\}$.

Proof. There exists all admissible URD($\{3,4\}; 324 \cdot 12 + 12 \equiv 65 \cdot 60$) by Theorem 5.8. There exists a $\{3,4\}$-URGDD of type 12^{27u+1} for $u \geq 5$, $u \neq 12$ and $r_4 \in \{4u, 4u + 2, \ldots, 108u\}$ by Lemma 4.34. The assertion follows by filling all groups with a URD($\{3,4\}$; 12) and by Theorem 1.11. □

Theorem 6.9. *There exist all admissible URD*($\{3,4\}; 360i + 192$) *for* $i \geq 2$.

Proof. There exists a $\{3,4\}$-URGDD of type 12^{30i+16} for $i \geq 2$ and $r_4 \in \{0, 2, 4, \ldots, 60(2i + 1)\}$ by Lemma 4.28. There exist all admissible URD($\{3,4\}; 360i + 192$) for $i \geq 2$ by filling in all groups with a URD($\{3,4\}$; 12) with $r_4 = 1$ and by Theorem 1.11. □

Lemma 6.10. *There exist all admissible URD*($\{3,4\};\ v$) *for* $v \equiv 120 + 144 \pmod{1008}$, $v \geq 5{,}304$.

Proof. There exists a $\{3,4\}$-frame of type $1{,}008^u$ for $u \geq 5$ and $\tilde{r}_4 \in \{0, 2, 4, \ldots, 336\}$ per group of the frame by Lemma 4.39. There exists a URD($\{3,4\}; 1{,}272$)$r_4 \in \{1, 3, 5, \ldots, 423\}$ by Theorem 6.9. There exists a IURD($\{3,4\}; 1{,}008 + 264$) with a hole of size 264 and $r_4 \in \{0, 2, 4, \ldots, 308, 336\}$, $r_4^0 \in \{1, 3, 5, \ldots, 85\}$ by Lemma 4.40. Adjoin 264 infinite points to the frame and fill $u - 1$ groups with the above IURD with the same r_4^0 but different r_4, where the infinite points fill the hole. Each group of the frame has to be filled with the same number of 4-pcs as are corresponding to the group. Then the number of 3-pcs corresponding to the group of the frame and its URGDD is also equal. We thus obtain $\hat{r}_4 \in \{0, 2, 4, \ldots, 336(u - 2) - 28, 336(u - 1)\}$ 4-pcs. The partial 4-pcs of all IURDs combine to form partial 4-pcs over all $u - 1$ groups, while in each case all r_4^0 are equal. We obtain $\hat{r}_4^0 \in \{1, 3, 5, \ldots, 85\}$ partial 4-pcs over all $u - 1$ groups. Together with the $\tilde{r}_4 \in \{0, 2, 4, \ldots, 336\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1, 3, 5, \ldots, 336 + 85\}$ partial 4-pcs over all $u - 1$ groups.

The URD($\{3,4\}; 1,272$) is used to fill in the last group together with the infinite points. We thus obtain $\breve{r}_4 \in \{1,3,5,\ldots,336+85\} \cap \{1,3,\ldots,423\} = \{1,3,\ldots,336+85\}$ 4-pcs. The result is a URD($\{3,4\}; 1,008\ u\ +264$) for $u \geq 5$ and $r_4 \in \{1,3,5,\ldots,336u+85\}$. We apply Theorem 1.11 for the greatest r_4. □

Lemma 6.11. *There exist all admissible URD*($\{3,4\};\ v$) *for* $v \in \{696, 1{,}704, 5{,}736\}$.

Proof. There exists an 8-RGDD of type 8^{7i+1} for $i \in \{4,10,34\}$ by [35], which we take as the master design. There exists a $\{3,4\}$-URGDD of type 3^8 with $r_4 \in \{1,3,5,7\}$ by Lemma 3.1. We expand all points of the master design three times and obtain a $\{3,4\}$-URGDD of type 24^{7i+1} with $r_4 \in \{8i, 8i+2,\ldots,56i\}$ for $i \in \{4,10,34\}$. The assertion follows by filling in all groups with the same URD($\{3,4\}; 24$) and by Theorem 6.1. □

For the last subclass $v \equiv 120 \pmod{144}$, we deal with v congruent 120, 264, and 408 modulo 432.

Lemma 6.12. *There exist all admissible URD*($\{3,4\};\ v$) *for* $v \equiv 408 \pmod{432}$, $v \geq 1,704$.

Proof. There exists a $\{4,6\}$-frame of type $(3;4^1)^{2(2i-1)}(5;6^1)^1$ for $i \geq 4$ and $i \neq 34$ by Lemma 4.43.

We take all $\{3,4\}$-URGDD of type 36^4 with $r_4 \in \{0,2,\ldots,36\}$ (Lemma 3.18) and 36^6 with $r_4 \in \{0,2,\ldots,54,60\}$ (Lemma 3.18) as ingredient designs. We expand all points of the frame 36 times and obtain a $\{3,4\}$-frame of type $108^{2(2i-1)}\,180^1$ with $\tilde{r}_4 \in \{0,2,4,\ldots,36\}$ per group of size 108 and $\breve{r}_4 \in \{0,2,\ldots,54,60\}$ per group of size 180.

There exists a $\{3,4\}$-URGDD of type 12^{10} with $r_4 \in \{0,12,36\}$ by Lemma 4.15. Adjoin 12 infinite points to the frame and fill all groups of size 108 with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0,12,24,\ldots,72(2i-1)-24,\ 72(2i-1)\}$ 4-pcs over all points.

We fill each new group of size 12 with a URD($\{3,4\}$; 12) with $r_4 = 1$ from Lemma 2.4, but not the infinite points. These URDs combine to form partial 4-pcs over all groups of size 108 with $r_4^0 = 1$. Together with the $\breve{r}_4 \in \{0,2,\ldots,54,60\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1,3,5,\ldots,55,61\}$ partial 4-pcs which miss exactly the points in the group of size 180.

The URD($\{3,4\}; 192$) (Theorem 1.14) with $r_4 \in \{1,3,\ldots,63\}$ is used to fill in the last group together with the infinite points. We thus obtain $\breve{r}_4 \in \{1,3,\ldots,55,61\}$ 4-pcs. The result is a URD($\{3,4\}; 216(2i-1)+192$) with $r_4 \in \{1,3,\ldots,72(2i-1)+55,\ 72(2i-1)+61\}$. The assertion follows by Theorems 1.11 and 1.13.

We now deal with the case $i = 34$ in a similar manner. There exists a 4-RGDD of type 24^{17} by Theorem 1.4. We remove a point and obtain a $\{4,24\}$-frame of type $(3;4^1)^{128}(23;24^1)^1$.

We take all $\{3,4\}$-URGDD of type 36^4 with $r_4 \in \{0,2,\ldots,36\}$ (Lemma 3.18) and 36^{24} with $r_4 \in \{0,2,\ldots,276\}$ (Lemma 3.23) as ingredient designs. We expand all points of the frame 36 times and obtain a $\{3,4\}$-frame of type $108^{128}\,828^1$ with $\tilde{r}_4 \in \{0,2,4,\ldots,36\}$ per group of size 108 and $\breve{r}_4 \in \{0,2,\ldots,276\}$ per group of size 828.

There exists a $\{3, 4\}$-URGDD of type 12^{10} with $r_4 \in \{0, 12, 36\}$ by Lemma 4.15. Adjoin 12 infinite points to the frame and fill all groups of size 108 with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0, 12, 24, \ldots, 4{,}608 - 24,\ 4{,}608\}$ 4-pcs over all points.

We fill each new group of size 12 with a URD($\{3, 4\}$; 12), but not the infinite points. These URDs combine to form partial 4-pcs over all groups of size 108 with $r_4^0 = 1$. Together with the $\breve{r}_4 \in \{0, 2, \ldots, 276\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1, 3, 5, \ldots, 277\}$ partial 4-pcs which miss exactly the points in the group of size 828.

The URD($\{3, 4\}$; 840) (Theorem 5.8) with $r_4 \in \{1, 3, \ldots, 279\}$ is used to fill in the last group together with the infinite points. We thus obtain $\overset{\frown}{r}_4 \in \{1, 3, \ldots, 277\}$ 4-pcs. The result is a URD($\{3, 4\}$; $13,824 + 840 \equiv 216 \cdot 67 + 192$) with $r_4 \in \{1, 3, \ldots, 4{,}608 + 277\}$. The assertion for this case follows by Theorems 1.11 and 1.13. □

Theorem 6.13. *There exist all admissible URD*($\{3, 4\}$; v) *for* $v \equiv 408 \pmod{432}$*, possibly excepting* $v = 408$*,* $r_4 \in \{121, 123, 125, 127\}$.

Proof. There exists a $\{3, 4\}$-URGDD of type 24^{17} with $r_4 \in \{0, 2, 4, \ldots, 112, 128\}$ by Lemma 4.36 and Theorem 1.4. We fill all groups with the same URD($\{3, 4\}$; 24) and obtain a URD($\{3, 4\}$; 408) with $r_4 \in \{1, 3, \ldots, 119, 129, 131, 133, 135\}$.

There exist all admissible URD($\{3, 4\}$; 840) by Theorem 5.8.

There exist all admissible URD($\{3, 4\}$; 1, 272) by Theorem 6.9.

The assertion follows by Lemma 6.12. □

Lemma 6.14. *There exist all admissible URD*($\{3, 4\}$; v) *for* $v \equiv 120 \pmod{432}$*, possibly excepting* $r_4 \in \{(v/3) - 13, (v/3) - 11, (v/3) - 9\}$.

Proof. The case $v = 120$ is handled in Lemma 3.9.

There exists a $\{3, 4\}$-frame of type 108^{4i+1} for $i \geq 1$ and $\tilde{r}_4 \in \{0, 2, 4, \ldots, 36\}$ per group of the frame by Lemma 4.37.

There exists a $\{3,4\}$-URGDD of type 12^{10} with $r_4 \in \{0, 12, 36\}$ by Lemma 4.15.

Adjoin 12 infinite points to the frame and fill $4i$ groups with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0, 12, 24, \ldots, 144i - 24,\ 144i\}$ 4-pcs.

We fill each new group of size 12 with a URD($\{3, 4\}$; 12), but not the infinite points. A URD($\{3, 4\}$; 120) (Lemma 3.9) with $r_4 \in \{1, 3, \ldots, 25, 33, 35, 37, 39\}$ is used to fill in the last group together with the infinite points. We thus obtain $\overset{\frown}{r}_4 \in \{1, 3, 5, \ldots, 37\} \cap \{1, 3, \ldots, 25, 33, 35, 37, 39\} = \{1, 3, \ldots, 25, 33, 35, 37\}$ 4-pcs. The result is a URD($\{3, 4\}$; $432\,i + 120$) for $i \geq 1$ with $r_4 \in \{1, 3, 5, \ldots, 144i + 25,\ 144i + 33,\ 144i + 35,\ 144i + 37\}$. We apply Theorem 1.11 for the greatest r_4. □

Lemma 6.15. *There exist all admissible URD*($\{3, 4\}$; v) *for* $v \equiv 120 \pmod{432}$*,* $v \geq 8,328$.

Proof. There exists a 5-GDD of type $(12i)^5(4j)^1$ for $i \geq 5$, $4j \leq (4/3) \cdot 12i = 16i$, i.e. $j \leq 4i$ by Theorem 1.2, which is our master design. We take a 4-frame of type 3^5 (Theorem 1.6) as ingredient design. We expand all points of the master design three times and obtain a 4-frame of type $(36i)^5\ (12j)^1$.

We take all $\{3,4\}$-URGDD of type 9^4 with $r_4 \in \{1,3,5,7,9\}$ (Lemma 2.3) as ingredient designs. We expand all points of the 4-frame nine times and obtain a $\{3,4\}$-frame of type $(324i)^5(108j)^1$ with $\tilde{r}_4 \in \{12i, 12i+2, 12i+4, \ldots, 108i\}$ per group of size $324i$ and $\breve{r}_4 \in \{4j, 4j+2, \ldots, 36j\}$ per group of size $108j$.

There exists a $\{3,4\}$-URGDD of type 12^{27i+1} with $r_4 \in \{4i, 4i+2, \ldots, 108i\}$ for $i \geq 5$, $i \neq 12$ by Lemma 4.34. Adjoin 12 infinite points to the frame and fill all groups of size $324i$ with one of the above URGDDs, where the infinite points form a group. We thus obtain $r_4' \in \{60i, 60i+2, 60i+4, \ldots, 540i\}$ 4-pcs which cover all points.

We fill each new group of size 12 with a URD($\{3,4\}$; 12), but not the infinite points. These URDs combine to form partial 4-pcs over all groups of size $324i$ with $r_4^0 = 1$. Together with the $\breve{r}_4 \in \{4j, 4j+2, \ldots, 36j\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{4j+1, 4j+3, \ldots, 36j+1\}$ partial 4-pcs which miss exactly the points of the group of size $108j$.

There exists a URD($\{3,4\}$; $108j+12$) with $r_4 = 36j+4-3 = 36j+1$ by Theorem 1.13. This URD is used to fill in the last group together with the infinite points. We thus obtain $\overset{\frown}{r}_4 = 36j+1$ 4-pcs, which adds to r_4' above.

Now let $v \equiv 120 \pmod{432}$, $v \geq 8{,}328$, and $i = \lfloor (v-120)/1{,}620 \rfloor$, then we have $i \geq 5$. The remainder $R = v - 120 - 1{,}620 \lfloor (v-120)/1{,}620 \rfloor \equiv 0 \pmod{108}$ and is smaller than 1,620. Let $j = 1 + (R/108)$, then we have $1 \leq j \leq 15 < 4i$. In particular, we have $v = 1{,}620i + 108j + 12$.

When $i \neq 12$, the result from above is a URD($\{3,4\}$; v) with $r_4 \in \{60i+36j+1, 60i+36j+3, \ldots, 540i+36j+1\}$. The assertion follows by Theorems 6.1 and 1.11.

In the case $i = 12$, there exists a $\{3,4\}$-URGDD of type 12^{27i+1} with $r_4 \in \{1{,}056, 1{,}058, \ldots, 1{,}296 = 108i\}$ by Lemma 4.15. The assertion for this case follows by Lemma 6.14. □

Theorem 6.16. *There exist all admissible* URD($\{3,4\}$; v) *for* $v \equiv 120 \pmod{432}$, *possibly excepting* $v \in \{120, 552, 984\}$, *and* $r_4 \in \{(v/3)-13, (v/3)-11, (v/3)-9\}$.

Proof. By Lemmas 6.14 and 6.15, there are 16 values to consider $v \in \{$1,416, 1,848, 2,280, 2,712, 3,144, 3,576, 4,008, 4,440, 4,872, 5,304, 5,736, 6,168, 6,600, 7,032, 7,464, 7,896$\}$.

For the case $v = 1{,}416$, there exists a 4-RGDD of type 4^{10} with $r_4 = 12$ by Theorem 1.4. We add the same point to each block of the first 4-pc, a second point to each block of the second 4-pc and so on. The result is a 5-GDD of type $4^{10}12^1$, which is our master design. We take a 4-frame of type 3^5 (Theorem 1.6) as ingredient design. We expand all points of the master design three times and obtain a 4-frame of type $12^{10}36^1$. We take all $\{3,4\}$-URGDD of type 9^4 with $r_4 \in \{1,3,5,7,9\}$ (Lemma 2.2) as ingredient designs. We expand all points of the 4-frame nine times and obtain a $\{3,4\}$-frame of type $108^{10}324^1$ with $\tilde{r}_4 \in \{4, 6, \ldots, 36\}$ per group of size 108 and $\breve{r}_4 \in \{12, 14, \ldots, 108\}$ per group of size 336. There exists a $\{3,4\}$-URGDD of type 12^{10} with $r_4 \in \{0, 12, 36\}$ by Lemma 4.15. Adjoin 12 infinite points to the frame and fill all groups of size 108 with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{120, 144, \ldots, 360\}$ 4-pcs over all points. We fill each new group of size 12 with a URD($\{3,4\}$; 12), but not the infinite points. These URDs combine to form partial 4-pcs over all groups of size 108 with $r_4^0 = 1$. Together with the $\breve{r}_4 \in \{12, 14, \ldots, 108\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{13, 15, \ldots, 109\}$ partial 4-pcs, which miss the group of size 324 and cover all of the points of the groups of size 108. The URD($\{3, 4\}; 336$) (Theorem 1.14) with $r_4 \in \{1, 3, \ldots, 111\}$ is used to fill in the last group together with the infinite points. We thus obtain $\overset{\frown}{r}_4 \in \{13, 15, \ldots, 109\}$ 4-pcs. The result is a URD($\{3, 4\}; 118 \cdot 12 = 1{,}416$) with $r_4 \in \{133, 135, \ldots, 360 + 109 = 469\}$. The assertion follows for this case by Theorems 1.11 and 6.1.

For the case $v = 1{,}848$, there exists a $\{3, 4\}$-URGDD of type 84^{22} with $r_4 \in \{28, 30, \ldots, 588\}$ by Lemma 4.23. We fill all groups with the same URD($\{3, 4\}; 84$) and obtain a URD($\{3, 4\}; 1, 848$) with $r_4 \in \{29, 31, \ldots, 615\}$. The assertion for this case follows by Theorem 6.1.

There exist all admissible URD($\{3, 4\}; 2{,}280$) by Theorem 5.8.

There exist all admissible URD($\{3, 4\}; 2{,}712$) by Theorem 6.9.

For the case $v = 3{,}144$, there exists a $\{6, 12\}$-frame of type $(5; 6^1)^{11}(11; 12^1)^1$ by Lemma 4.44. We take all $\{3, 4\}$-URGDD of type 48^6 with $r_4 \in \{0, 2, \ldots, 72, 80\}$ (Lemma 4.42) and 48^{11} with $r_4 \in \{0, 2, 4, \ldots, 160\}$ (Lemma 4.42) as ingredient designs. We expand all points of the frame 48 times and obtain a $\{3, 4\}$-frame of type $240^{11}\, 528^1$ with $\tilde{r}_4 \in \{0, 2, \ldots, 72, 80\}$ per group of size 240 and $\breve{r}_4 \in \{0, 2, \ldots, 160\}$ per group of size 528. There exists a $\{3, 4\}$-URGDD of type 24^{11} with $r_4 \in \{0, 2, \ldots, 70, 80\}$ by Lemma 4.36. Adjoin 24 infinite points to the frame and fill all groups of size 240 with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0, 2, \ldots, 870, 880\}$ 4-pcs over all points. We fill each new group of size 24 with the same URD($\{3, 4\}; 24$), but not the infinite points. These URDs combine to form partial 4-pcs over all groups of size 240 with $r_4^0 \in \{1, 3, \ldots, 7\}$. Together with the $\breve{r}_4 \in \{0, 2, \ldots, 160\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1, 3, 5, \ldots, 167\}$ partial 4-pcs, which miss the group of size 528 and cover all of the points of the groups of size 240.

A URD($\{3, 4\}; 504$) with $r_4 \in \{1, 3, \ldots, 167\}$ (Theorem 6.7) is used to fill in the last group together with the infinite points. We thus obtain $\overset{\frown}{r}_4 \in \{1, 3, \ldots, 167\}$ 4-pcs. The result is a URD($\{3, 4\}; 262 \cdot 12 = 3, 144$) with $r_4 \in \{1, 3, \ldots, 1{,}047\}$.

There exist all admissible URD($\{3, 4\}; 3, 576$) by Theorem 6.1 and Lemma 6.8.

For the case $v = 4{,}008$, there exists a 5-GDD of type $16^5 8^1$ by Theorem 1.2, which is our master design. We take a 4-frame of type 3^5 (Theorem 1.6) as ingredient design. We expand all points of the master design three times and obtain a 4-frame of type $48^5 24^1$. We take all $\{3, 4\}$-URGDD of type 15^4 with $r_4 \in \{1, 3, \ldots, 15\}$ (Lemma 2.5) as ingredient designs. We expand all points of the 4-frame 15 times and obtain a $\{3, 4\}$-frame of type $720^5 360^1$ with $\tilde{r}_4 \in \{16, 18, \ldots, 240\}$ per group of size 720 and $\breve{r}_4 \in \{8, 10, \ldots, 120\}$ per group of size 360. There exists a $\{3, 4\}$-URGDD of type 48^{16} with $r_4 \in \{0, 2, 4, \ldots, 240\}$ by Lemma 4.41. Adjoin 48 infinite points to the frame and fill all groups of size 720 with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{80, 82, \ldots, 1{,}200\}$ 4-pcs over all points. We fill each new group of size 48 with the same URD($\{3, 4\}; 48$), but not the infinite points. These URDs combine to form partial 4-pcs over all five groups of size 720 with $r_4^0 \in \{1, 3, \ldots, 15\}$. Together with the $\breve{r}_4 \in \{8, 10, \ldots, 120\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{9, 11, \ldots, 135\}$ partial 4-pcs partial 4-pcs which miss the group of size 360 and cover all of the points of the groups of size 720. There exists a $\{3, 4\}$-URGDD of type 24^{17} with $r_4 \in \{16, 18, \ldots, 112\}$ by Lemma 4.36. By filling the groups, Theorems 6.1 and 1.13, we obtain a URD($\{3, 4\}; 408$) with $r_4 \in \{1, 3, \ldots, 119, 129, 131, 133, 135\}$, which is used to fill in the last group together with the infinite points. We thus obtain $\overset{\frown}{r}_4 \in \{9, 11, \ldots, 119, 129, 131, 133, 135\}$ 4-pcs.

The result is a URD($\{3, 4\}; 324 \cdot 12 = 4{,}008$)with $r_4 \in \{89, 91, \dots, 1{,}335\}$. The assertion for this case follows by Theorem 6.1.

There exist all admissible URD($\{3, 4\}; 4{,}440$) by Theorem 5.8.
There exist all admissible URD($\{3, 4\}; 4{,}872$) by Theorem 6.9.
There exist all admissible URD($\{3, 4\}; 5{,}304$) by Lemma 6.10.
There exist all admissible URD($\{3, 4\}; 5{,}736$) by Lemma 6.11.
There exist all admissible URD($\{3, 4\}; 6{,}168$) by Theorem 6.1 and Lemma 6.8.
There exist all admissible URD($\{3, 4\}; 6{,}600$) by Theorem 5.8.
There exist all admissible URD($\{3, 4\}; 7{,}032$) by Theorem 6.9.

For the case $v = 7{,}464$, there exists a 4-RGDD of type 8^{13} with $r_4^0 = 32$ by Theorem 1.4, which we take as the master design. We take the URGDDs of Lemma 2.5 as ingredient designs. We expand all points of the master design 15 times. We obtain a $\{3, 4\}$-URGDD of type 120^{13} with $r_4 \in \{32, 34, 36, \dots, 480\}$, as we fill all parallel classes appropriately. There exists a 5-GDD of type $40^5 4^1$ by Theorem 1.2, which is our master design. We take a 4-frame of type 3^5 (Theorem 1.6) as ingredient design. We expand all points of the master design three times and obtain a 4-frame of type$120^5 12^1$. We take all $\{3, 4\}$-URGDD of type 12^4 with $r_4 \in \{0, 2, \dots, 12\}$ (Lemma 2.3) as ingredient designs. We expand all points of the 4-frame 12 times and obtain a $\{3, 4\}$-frame of type $(120 \cdot 12)^5 144^1$ with $\tilde{r}_4 \in \{0, 2, \dots, 480\}$ per group of size 1,440 and $\breve{r}_4 \in \{0, 2, \dots, 48\}$ per group of size 144. We take from above a $\{3, 4\}$-URGDD of type 120^{13} with $r_4 \in \{32, 34, 36, \dots, 480\}$. Adjoin 120 infinite points to the frame and fill all groups of size 120 with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{160, 162, 164, \dots, 2{,}400\}$ 4-pcs over all points.

We fill each new group of size 120 with the same URD($\{3, 4\}; 120$), but not the infinite points. These URDs combine to form partial 4-pcs over all groups of size 1,440 with $r_4^0 \in \{1, 3, \dots, 25, 33, 35, 37, 39\}$. Together with the $\breve{r}_4 \in \{0, 2, \dots, 48\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1, 3, 5, \dots, 87\}$ partial 4-pcs which miss the group of size 144 and cover all of the points contained in groups of size 1,440. A URD($\{3, 4\}; 264$) with $r_4 \in \{1, 3, \dots, 77, 81, 83, 85, 87\}$(next Lemma) is used to fill in the last group together with the infinite points. We thus obtain $\overset{\frown}{r}_4 \in \{1, 3, \dots, 77, 81, 83, 85, 87\}$ 4-pcs. The result is a URD($\{3, 4\}; 622 \cdot 12 = 7{,}464$) with $r_4 \in \{161, 163, \dots, 2{,}487\}$. The assertion for this case follows by Theorem 6.1.

For the final case $v = 7{,}896$, there exists a $\{3, 4\}$-URGDD of type 84^{94} with $r_4 \in \{124, 126, \dots, 2{,}604\}$ by Lemma 4.23. We fill all groups with the same URD($\{3, 4\}; 84$) and obtain a URD($\{3, 4\}; 7{,}896$) with $r_4 \in \{125, 127, \dots, 2{,}631\}$. The assertion for this case follows by Theorem 6.1. □

For the last subsubclass $v \equiv 264 \pmod{432}$, we deal with v congruent 264, 696, 1,128, 1,560, and 1,992 modulo 2160.

Lemma 6.17. *There exist all admissible URD($\{3, 4\}$; v) for $v \equiv 264 \pmod{2160}$, possibly excepting $v = 264$, $r_4 = 79$.*

Proof. There exists a $\{3, 4\}$-URGDD of type 24^{11} with $r_4 \in \{0, 2, 4, \dots, 70\}$ by Lemma 4.36. Filling all groups appropriately with the same URD($\{3, 4\}; 24$) results in a URD($\{3, 4\}; 264$) with $r_4 \in \{1, 3, \dots, 77\}$. We obtain $r_4 \in \{1, 3, \dots, 77, 81, 83, 85, 87\}$ for this design by Theorems 1.11 and 1.13.

There exists a 5-GDD of type $(8i)^5 4^1$ for $i \geq 1$ by Theorem 1.2, which is our master design. We take a 4-frame of type 3^5 (Theorem 1.6) as ingredient design. We expand all points of the master design three times and obtain a 4-frame of type $(24i)^5 12^1$.

We take all $\{3, 4\}$-URGDD of type 18^4 with $r_4 \in \{0, 2, \ldots, 18\}$ (Lemma 2.6) as ingredient designs. We expand all points of the 4-frame 18 times and obtain a $\{3, 4\}$-frame of type $(432i)^5 216^1$ with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 144i\}$ per group of size $432i$ and $\breve{r}_4 \in \{0, 2, \ldots, 72\}$ per group of size 216.

There exists a $\{3, 4\}$-URGDD of type 48^{9i+1} with $r_4 \in \{0, 2, \ldots, 144i\}$ for $i \geq 1$ by Lemma 4.41. Adjoin 48 infinite points to the frame and fill all groups of size $432i$ with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0, 2, 4, \ldots, 720i\}$ 4-pcs over all points.

We fill each new group of size 48 with the same URD($\{3, 4\}; 48$), but not the infinite points. These URDs combine to form partial 4-pcs over all five groups of size $432i$ with $r_4^0 \in \{1, 3, \ldots, 15\}$. Together with the $\breve{r}_4 \in \{0, 2, \ldots, 72\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1, 3, 5, \ldots, 87\}$ partial 4-pcs which miss the group of size 216 and cover all of the points of the groups of size $432i$.

The URD($\{3, 4\}; 264$) with $r_4 \in \{1, 3, \ldots, 77, 81, 83, 85, 87\}$ from above is used to fill in the last group together with the infinite points. We thus obtain $\frown{r}_4 \in \{1, 3, \ldots, 77, 81, 83, 85, 87\}$ 4-pcs. The result is a URD($\{3, 4\}; 2,160i + 264$) with $r_4 \in \{1, 3, \ldots, 720i + 87\}$ for $i \geq 1$. □

Lemma 6.18. *There exist all admissible URD*($\{3, 4\}$; v) *for* $v \equiv 696 \pmod{720}$, $v \geq 2{,}856$.

Proof. There exists a $\{4, 6\}$-frame of type $(3; 4^1)^{2(2i-1)}(5; 6^1)^1$ for $i \geq 4$ and $i \neq 34$ by Lemma 4.43.

We take all $\{3, 4\}$-URGDD of type 60^4 with $r_4 \in \{0, 2, \ldots, 60\}$ (Lemma 2.10) and 60^6 with $r_4 \in \{0, 2, \ldots, 90, 100\}$ (Lemma 4.16) as ingredient designs. We expand all points of the frame 60 times and obtain a $\{3, 4\}$-frame of type $180^{2(2i-1)} 300^1$ with $\tilde{r}_4 \in \{0, 2, 4, \ldots, 60\}$ per group of size 180 and $\breve{r}_4 \in \{0, 2, \ldots, 90, 100\}$ per group of size 300.

There exists a $\{3, 4\}$-URGDD of type 36^6 with $r_4 \in \{0, 2, \ldots, 54, 60\}$ by Lemma 3.18. Adjoin 36 infinite points to the frame and fill all groups of size 180 with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0, 2, 4, \ldots, 120(2i-1) - 6, 120(2i-1)\}$ 4-pcs over all points.

We fill each new group of size 36 with the same URD($\{3, 4\}; 36$), but not the infinite points. These URDs combine to form partial 4-pcs over all groups of size 180 with $r_4^0 \in \{1, 3, \ldots, 11\}$. Together with the $\breve{r}_4 \in \{0, 2, \ldots, 90, 100\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1, 3, 5, \ldots, 111\}$ partial 4-pcs which miss the group of size 300 and cover all of the points of the groups of size 180.

A URD($\{3, 4\}; 336$) (Theorem 1.14) with $r_4 \in \{1, 3, \ldots, 111\}$ is used to fill in the last group together with the infinite points. We thus obtain $\frown{r}_4 \in \{1, 3, \ldots, 111\}$ 4-pcs. The result is a URD($\{3, 4\}; 360(2i-1) + 336$) with $r_4 \in \{1, 3, \ldots, 120(2i-1) + 111\}$ for $i \geq 4$ and $i \neq 34$.

Now the case $i = 34$. There exists a 4-RGDD of type 24^{17} by Theorem 1.4. We remove a point and obtain a $\{4, 24\}$-frame of type $(3; 4^1)^{128}(23; 24^1)^1$.

We take all $\{3, 4\}$-URGDD of type 60^4 (Lemma 2.10) with $r_4 \in \{0, 2, \ldots, 60\}$ and 60^{24} with $r_4 \in \{0, 2, \ldots, 460\}$ (Lemma 4.16) as ingredient designs. We expand all points of the frame 60 times and obtain a $\{3,4\}$-frame of type $180^{128}\,1,380^1$ with $\tilde{r}_4 \in \{0,2,4,\ldots,60\}$ per group of size 180 and $\breve{r}_4 \in \{0,2,\ldots,460\}$ per group of size 1,380.

There exists a $\{3,4\}$-URGDD of type 36^6 with $r_4 \in \{0,2,\ldots,54,60\}$ by Lemma 3.18. Adjoin 36 infinite points to the frame and fill all groups of size 180 with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0,2,4,\ldots,60\cdot 128-6, 60\cdot 128\}$ 4-pcs over all points.

We fill each new group of size 36 with the same URD($\{3,4\}$; 36), but not the infinite points. These URDs combine to form partial 4-pcs over all groups of size 180 with $r_4^0 \in \{1,3,\ldots,11\}$. Together with the $\breve{r}_4 \in \{0,2,\ldots,460\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1,3,5,\ldots,471\}$ partial 4-pcs which miss the group of size 1,380 and cover all of the points of the groups of size 180.

A URD($\{3,4\}$; 1,416) (Lemma 6.14) with $r_4 \in \{1,3,\ldots,457,465,467,469,471\}$ is used to fill in the last group together with the infinite points. We thus obtain $\overset{\frown}{r}_4 \in \{1,3,\ldots,457,465,467,469,471\}$ 4-pcs. The result is a URD($\{3,4\}$; $180\cdot 128 + 1,416 = 180\cdot 134 + 336$) with $r_4 \in \{1,3,\ldots,8,151\}$. □

Corollary 6.19. *There exist all admissible URD*($\{3,4\}$; v) *for* $v \equiv 696 \pmod{2,160}$.

Proof. There exist all admissible URD($\{3,4\}$; 696) by Lemma 6.11. The assertion follows by Lemma 6.18. □

Lemma 6.20. *There exist all admissible URD*($\{3,4\}$; v) *for* $v \equiv 1, 128 \pmod{2160}$, $v \geq 5,448$.

Proof. There exists a 5-GDD of type $(8i)^5 20^1$ for $i \geq 2$ by Theorem 1.2, which is our master design. We take a 4-frame of type 3^5 (Theorem 1.6) as ingredient design. We expand all points of the master design three times and obtain a 4-frame of type$(24i)^5 60^1$.

We take all $\{3,4\}$-URGDD of type 18^4 with $r_4 \in \{0,2,\ldots,18\}$ (Lemma 2.6) as ingredient designs. We expand all points of the 4-frame 18 times and obtain a $\{3,4\}$-frame of type $(432i)^5 1{,}080^1$ with $\tilde{r}_4 \in \{0,2,4,\ldots,144i\}$ per group of size $432i$ and $\breve{r}_4 \in \{0,2,\ldots,360\}$ per group of size 1,080.

There exists a $\{3,4\}$-URGDD of type 48^{9i+1} with $r_4 \in \{0,2,\ldots,144i\}$ by Lemma 4.41. Adjoin 48 infinite points to the frame and fill all groups of size $432i$ with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0,2,4,\ldots,720i\}$ 4-pcs over all points.

We fill each new group of size 48 with the same URD($\{3,4\}$; 48), but not the infinite points. These URDs combine to form partial 4-pcs over all five groups of size $432i$ with $r_4^0 \in \{1,3,\ldots,15\}$. Together with the $\breve{r}_4 \in \{0,2,\ldots,360\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1,3,5,\ldots,375\}$ partial 4-pcs, which miss the group of size 1,080 and cover all of the points of the groups of size $432i$.

A URD($\{3,4\}$; 1,128) with $r_4 \in \{1,3,\ldots,9,369,371,373,375\}$ (Theorems 1.12 and 1.13) is used to fill in the last group together with the infinite points. We thus obtain $\overset{\frown}{r}_4 \in \{1,3,\ldots,9,369,371,373,375\}$ 4-pcs. The result is a URD($\{3,4\}$; $2{,}160i + 1{,}128$) with $r_4 \in \{1,3,\ldots,720i+375\}$ for $i \geq 2$. □

Corollary 6.21. *There exist all admissible URD*($\{3,4\}$; v) *for* $v \equiv 1{,}560 \pmod{2160}$.

Proof. The assertion follows by Theorem 5.8. □

Corollary 6.22. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 1{,}992 \pmod{2160}$.

Proof. The assertion follows by Theorem 6.9 with $i \equiv 5 \pmod 6$. □

Theorem 6.23. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 264 \pmod{432}$, *possibly excepting* $v \in \{264, 1{,}128, 3{,}288\}$, $r_4 = (v/3) - 9$.

Proof. Lemma 6.17, Corollary 6.19, Lemma 6.20, Corollary 6.21, and Corollary 6.22 cover every case except $v \in \{1{,}128, 3{,}288\}$.

For the case $v = 1{,}128$, there exists a 4-RGDD of type 8^4 by Theorem 1.4. We remove a point and obtain a $\{4, 8\}$-frame of type $(3; 4^1)^8(7; 8^1)^1$.

We take all $\{3, 4\}$-URGDD of type 36^4 with $r_4 \in \{0, 2, \dots, 36\}$ (Lemma 3.18) and 36^8 with $r_4 \in \{0, 2, \dots, 84\}$ (Lemma 3.23) as ingredient designs. We expand all points of the frame 36 times and obtain a $\{3, 4\}$-frame of type $108^8\,252^1$ with $\tilde{r}_4 \in \{0, 2, 4, \dots, 36\}$ per group of size 108 and $\breve{r}_4 \in \{0, 2, \dots, 84\}$ per group of size 252. There exists a $\{3, 4\}$-URGDD of type 12^{10} with $r_4 \in \{0, 12, 36\}$ by Lemma 4.15. Adjoin 12 infinite points to the frame and fill all groups of size 108 with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0, 12, 24, \dots, 264,\ 288\}$ 4-pcs over all points. We fill each new group of size 12 with a URD$(\{3, 4\}; 12)$, but not the infinite points. These URDs combine to form partial 4-pcs over all groups of size 108 with $r_4^0 = 1$. Together with the $\breve{r}_4 \in \{0, 2, \dots, 84\}$ partial 4-pcs of the last group, we obtain $\tilde{r}_4^0 \in \{1, 3, 5, \dots, 85\}$ partial 4-pcs which miss the group of size 252 and cover all of the points contained in groups of size 108. A URD$(\{3, 4\}; 264)$ (Lemma 6.17) with $r_4 \in \{1, 3, \dots, 77, 81, 83, 85, 87\}$ is used to fill in the last group together with the infinite points. We thus obtain $\widehat{r}_4 \in \{1, 3, \dots, 77, 81, 83, 85\}$ 4-pcs. The result is a URD$(\{3, 4\}; 1, 128)$ with $r_4 \in \{1, 3, \dots, 288 + 77,\ \ 288 + 81, 288 + 83, 288 + 85\}$. The assertion for this case follows by Theorem 1.11.

For the case $v = 3{,}288$, there exists a $\{3, 4\}$-frame of type 252^{13} with $\tilde{r}_4 \in \{0, 2, 4, \dots, 84\}$ per group of the frame by Lemma 4.38. There exists a $\{3, 4\}$-URGDD of type 12^{21+1} with $r_4 \in \{0, 2, 4, \dots, 56, 84\}$ by Lemma 4.26. Adjoin 12 infinite points to the frame and fill 12 groups with one of the above URGDDs, where the infinite points form a group. We thus obtain $\hat{r}_4 \in \{0, 2, 4, \dots, 980, 1{,}008\}$ 4-pcs. We fill each new group of size 12 with a URD$(\{3, 4\}; 12)$, but not the infinite points. A URD$(\{3, 4\}; 264)$ (Lemma 6.17) with $r_4 \in \{1, 3, \dots, 77, 81, 83, 85, 87\}$ is used to fill in the last group together with the infinite points. We thus obtain $\widehat{r}_4 \in \{1, 3, 5, \dots, 85\} \cap \{1, 3, \dots, 77, 81, 83, 85, 87\} = \{1, 3, \dots, 77, 81, 83, 85\}$ 4-pcs. The result is a URD$(\{3, 4\}; 12 \cdot 252\ + 264 = 3{,}288)$ with $r_4 \in \{1, 3, 5, \dots, 1{,}085,\ 1{,}089, 1{,}091, 1{,}093\}$. We apply Theorem 1.11 for the greatest r_4. □

We summarize the results of this section.

Theorem 6.24. *There exist all admissible URD*$(\{3, 4\};\ v)$ *for* $v \equiv 24 \pmod{48}$, *possibly excepting*

$v = 120$ *and* $r_4 \in \{(v/3) - 13, (v/3) - 11, (v/3) - 9\}$;

$v = 264$ *and* $r_4 = (v/3) - 9$;

$v = 408$ *and* $r_4 \in \{(v/3) - 15,\ (v/3) - 13, (v/3) - 11, (v/3) - 9\}$;

$v = 456$ *and* $r_4 \in \{(v/3) - 11, (v/3) - 9\}$;
$v = 552$ *and* $r_4 \in \{(v/3) - 13, (v/3) - 11, (v/3) - 9\}$;
$v = 984$ *and* $r_4 \in \{(v/3) - 13, (v/3) - 11, (v/3) - 9\}$;
$v = 1{,}128$ *and* $r_4 = (v/3) - 9$;
$v = 3{,}288$ *and* $r_4 = (v/3) - 9$.

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